A dynamic active learning trajectory for the construction of number pi (π): transforming mathematics education

Dr. Stavroula Patsiomitou[1]

[1]*Department of Mathematics and Applied Mathematics, University of Crete, Greece*
spatsiom@gmail.com

ABSTRACT

In the current study, I shall describe characteristic elements of a 'dynamic' trajectory, a synthesis of experts of my previous research regarding an approximation process for the construction of number pi, enriched, analyzed and reviewed in the light of my recent theoretical considerations (not published as yet). Number pi is a mathematical abstract object but it can also be perceived as a result of a process. Specific examples from my experimental research using dynamic active representations will be analyzed in the methodology section. Moreover, a brief report of students (small groups or individuals) which participated in the process. My aim was the students to conceive the meaning of number pi as a limit using the iteration process of the Geometer's Sketchpad dynamic geometry software. Finally, the role the active representations play in the learning trajectory made me think of a way to define what a 'dynamic active learning trajectory' is.

Key words: dynamic active learning trajectory, active representations, number pi, dynamic geometry

INTRODUCTION

In the present study, I shall describe characteristic elements of a 'dynamic' trajectory (i.e. a learning path through a dynamic geometry software) for the learning of concept of 'pi' (π) in mathematics. The paper highlights the importance of orchestrating different digital tools or static means into a learning trajectory, which helps students, develop their understanding of the concept of pi as an estimating process and use

> "the interactive representational affordances of technology (visualization, linking representations to each other and to simulations, importing physical data into the mathematical realm in active ways, graphically editing piecewise –defined functions, etc.) […] that lay the base for more advanced mathematics, particularly calculus (Hegedus & Kaput, 2004, p.129).

The inspiration for the construction of the meaning of pi occurred in class when I was teaching my students (aged 13 during the 2005–06 school year). I was using the images in a textbook to try to explain to them how the circumference of a circle tends to coincide with the perimeter of a regular 60-gon (i.e. polygon with 60 sides) inscribed in the circle with n increasing. The students—especially the students with low level achievement in mathematics—could not understand the difference between the image of the 60-gon and the image of a circle, as visually they had the same illustration in their textbook. The image of the 60-gon caused a cognitive conflict for most of my students. Since this misunderstanding prevented me from proceeding any further with my students, I decided to change my instructional path and explain the same things through a story of how Archimedes (287–212 BC) might have thought. Archimedes method of exhaustion was the first theoretical calculation of number pi (π). He conceived this method and obtained the approximation $^{223}/_{71} < \pi < {}^{22}/_{7}$. As we know, Archimedes calculated the perimeters of inscribed and circumscribed regular polygons of a circle whose perimeters converged to the circumference in an attempt to find the ratio between the circumference and the diameter.

I started my instructional approach as follows, creating figures on the blackboard in cooperation with my students:

> **Researcher:** *Can you construct a regular hexagon inscribed within a circle? Can you now construct a regular 12-gon in the same circle*?

The students created a circle and then an inscribed hexagon. They then constructed lines perpendicular to the sides of the hexagon to construct the 12-gon. At this point, I activated my students' pre-existing knowledge.

Researcher: *What do you observe*? *What can you say about the length of the sides of the 12-gon*?

The students answered that the length of the sides had decreased while the number of sides had increased.

Researcher: *Archimedes possibly thought this method to discover pi. He constructed a circle and then constructed polygons inscribed within the circle*. *Where do we stop*? *Can we increase the number of sides continuously*?

The students went on to construct a 24-gon, at which point a few of them stopped and noted: "*We need a larger circle to construct a 48-gon*". I asked them to measure the length of the sides of each of the polygons they had constructed, and then to calculate the ratio of the perimeter of the polygon to the diameter of the circle. They started to formulate conjectures, but many students in the class could not follow the discussion and did not seem to have a conception of the meanings.

This students' understanding completely modified, when I used predesigned activities or activities that I prepared for the lesson or constructed during the lesson in class in the Geometer's Sketchpad (Jackiw, 2001) dynamic geometry software (DGS). The children were able to visualize the image of the 60-gon when I enlarged it using the zoom tool preference provided by the software. Now, most of them could answer my questions. This brought the realization that a synthesis of different supporting technological and computing "ingredients"—such as a dynamic geometry environment and other technological digital resources—in everyday lessons could enhance the development of students' understanding in mathematics.

In recent years, my year high school students has reinvented the concept of 'pi' by following the same learning trajectory I described briefly above, which was "well-considered and empirically grounded" (Gravemejer, 2004, p.110). The learning trajectory combined history of mathematics, algebraic formulas, geometry, calculus and technology and was addressed primarily at high school students aged 13–16.

In the current study, I shall be describing: (a) the construction of number pi as an approximation process (Patsiomitou, 2006c, 2007a, 2009). For this, I created in the Geometer's Sketchpad software the process of an inscribed or circumscribed n-gon in a circle with a view to using the tabularized measurements and calculations of a ratio in combination with the software's iteration process to lead the students to visualize the approximation process of number pi; (b) the construction of number pi through Riemann sums in a DGS environment (Patsiomitou, 2006c); (c) the construction of number pi by means of a real world problem (Patsiomitou, 2013b, 2016a, b). Concretely, I combined a digital visit to the Guggenheim museum in New York using the Google Earth software (Website, [1]) with dynamic representations of the Geometer's Sketchpad software (Patsiomitou, 2016a, b) and other digital web resources.

I shall also be describing three parts of the experimental process: (a) For the first process, an excerpt from a discussion with a group of students. The students' formulations, their reactions as they interacted with the software as well as their interpretations and their connection of meanings were repeated with a detailed accuracy almost the same every year. (b) For the second process, a case study with a 13 year-old student in 2005 gave me to understand that we can teach pre-calculus formal mathematics to young learners if we use the proper language (e.g., Bruner, 1966; Fuys et al., 1984). The language we use to teach students is very important; if the students can make sense of the meaning we teach them, then they have the necessary "step" needed for the understanding in formal deductive reasoning. (c) For the third approach, I captured linking visual images with Google Earth software and elaborated/processed them using the Geometer's Sketchpad dynamic geometry software. The linking multiple representations helped the participating students to understand the meanings. There is a good deal of research which supports the hypothesis that the competence to translate between several modes of representation is connected with the students' ability to understand mathematical concepts (e.g., Sfard, 1992; Yerushalmy, 1997).

Following Sfard (1991) who defines "*learning as the process of changing ones discursive ways in a certain well-defined manner*" (p. 4) I tried to "*change the way my students think, and the issue of how we communicate this thinking*" (Sfard, 1991, p.4). Number pi is a mathematical abstract object but it can be also perceived as a result of a process as Sfard (1991) supports the idea of the "dual nature of mathematical representations of concepts", perceived as either objects or processes. Moreover, a geometrical diagram can be considered as a metaphor for the corresponding algebraic formula (Johnson, 1987). Sfard (1994) interpreted the meaning of a metaphor in the sense of Lakoff & Nùñez (2000), reporting that a metaphor is "a mental construction which plays a constitutive role, in structuring our experience and in shaping our imagination and reasoning" (p. 46).

During a problem-solving process in a DGS environment, students can discover a particular new piece of mathematical knowledge; for example, they can discover a meaning by simple logical deduction or they can see "geometry as just an accumulation of empirically discovered facts […]" (Jones, 2002, p.132), using the dynamic the representations effectively for mathematical understanding. Furthermore, *dynamic reinvention of knowledge* is the kind of knowledge the students could reinvent by interacting with the artefacts made in a DGS environment (Patsiomitou, 2012b, p. 57), "knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p.786).

Collaborating with children for the construction of meanings along a learning trajectory which includes static and dynamic means, we have to take under consideration the *instrumental approach* (e.g. Trouche, 2004; Trouche, Drijvers, 2014), which is rooted in cognitive ergonomics (Verillon & Rabardel, 1995), as well as the anthropological theory of didactics (Chevallard, 1999). This approach points out the interaction between a tool and its user, while the notion of *instrumental orchestration* developed by Trouche (2004) "*highlights the importance of the teacher in the students' process*" (Trouche & Drijvers, 2014, p.1). The role of the teacher as mediator is crucial during the active process of teaching and learning mathematics while according to Mariotti (2002, p.772) "*the construction of meanings occurs as the product of a process of internalization guided by the teacher*". Points of departure for the anticipation of the *instrumental approach* though the dynamic trajectories were the questions:

- How could a math lesson acquire interest for all students? Can external linking multiple representations captured by a digital medium help students to link concepts and meanings across different disciplines such as geography, mathematics and history of mathematics?
- Do students understand the mathematical components of a mathematical meaning when they see real-world images?
- Can these linking images help students to recall important information which it is difficult to recall under other circumstances?
- What mathematical activities are most reflective of, and appropriate for, the essential development of students' logico-mathematical structures?
- How important is the role of a dynamic geometry software in reorganizing students' mental representations?
- How effective is the teaching and learning process that uses linking visual active representations to overcome cognitive or instrumental obstacles and develop students' understanding of mathematical concepts?
- How does instruction conducted through technological tools and explorations within a laboratory environment help high school math students to discover mathematics?

The sections that follow will present the following:

(a) A review of the theoretical underpinnings, underlying concepts and definitions used for analyzing results in order to make visible the spectrum of possibilities at both the theoretical and methodological levels.
(b) Specific examples from my experimental research using dynamic representations will be analyzed in the methodology section.
(c) A brief report of students (small groups or individuals) which participated, working or implementing the approaching process in static or dynamic means, plus a final report on the results.

In the next section I shall present a theoretical underpinning, starting with the meaning of dynamic linking visual active representations (e.g., Patsiomitou, 2007b, 2008a, b, 2010, 2012a, b).

DYNAMIC LINKING VISUAL ACTIVE REPRESENTATIONS

Before analyzing the term "*representation*" we have to consider what Francis Crick (1994) argues: "[…] *the brain does not passively record the incoming visual information. It actively seeks to interpret it*". (p. 31). Moreover, the brain actively seeks to connect incoming visual information with previously stored knowledge created from information previously processed in the learner's mind. Tchoshanov (2013) considers that

> "[…] representations may be thought of as external stimuli (numerals, equations, graphs, tables, diagrams, etc.) of concepts or internal cognitive schemata –abstractions of ideas that are developed by a learner through experience. Representation could also refer to the act of externalizing an internal, mental abstraction. The key question is the relationship between external and internal representations in learning how students' mental schemata assimilates external representations, and how new external representations help students to accommodate their emerging internal representations" (pp.73-74).

Students' understanding of meanings often led me to note the sequence of steps or stages through which they gathered information from the [computing] environment as stimuli. The information from the computer environment goes through a modification, linked to students' minds stored information (or is modified in the light of the information stored in their mind) so they can answer the teacher's questions or participate in a class discussion. The students possibly transform this stimulus into mental representations linked with similar pre-existing information in their minds. I say 'possibly', as I cannot see into my students' minds. What I can see is the following: those representations that could not be linked to previous information were rejected by the students, as they did not answer. What is important to investigate is the level of strength of these links or connections in the students' mind, which can illustrate how the learning of meanings was accomplished.

As Thomas (2004, p.14) states

> "we might say that because conceptual ideas may be constructed from a number of representations it is a good idea for students to experience a number of these at the time they begin to learn the concept. In particular, the explicit linking of ideas across representations is very useful and important. Hence, teaching should seek to assist

> representational versatility by concurrently providing, and linking, a number of representations in each learning situation."

In order to develop an understanding of a meaning, the students have to create a transitional bridge between the external and internal representation (e.g, Kaput, 1999; Goldin & Shteingold, 2001; Pape & Tchoshanov, 2001) of this meaning. Moreover, students' visualization of an object may differ from their perception of it, while the important thing is to understand which mathematical concept or relationship is being represented. Visual representation systems encourage students to interact with visually represented mathematical concepts and ideas. Since the 1980s, computers and technological tools have changed the way students, teachers and researchers think, allowing them to visualize mathematical ideas in ways that were previously impossible (Kaput, 1992; Balachef and Kaput, 1997). Technological tools can help the students focus their attention and translate between mathematical representations or interpret information received from a real world environment. Computers and technological tools have changed the context of mathematical activity, imbuing the instructional design process, and the relationship between math and other contexts, with new possibilities. Pea (1987) also supports that these tools

> "help students develop the languages of mathematical thought by linking different representations of mathematical concepts, relationships, and processes.[…] The languages of mathematical thought, which become apparent in these different representations, include: (a) Natural language description of mathematical relations (e.g., linear equations); (b) Equations composed of mathematical symbols (e.g., linear equations); (c) Visual Cartesian coordinate graphs of functions in two and three dimensions; (d) Graphic representations of objects (e.g., in place-value subtraction, the use of "bins" of objects representing different types of place units"(p.109)

The different use of computing environments in education has led to several classifications. Papert (1984) in his paper "Microworlds: transforming education" considers three kinds of computer use: as tutorials, as tools and as microworlds. He concentrated on the notion of the microworld "in the sense that it's a little world, a little slice of reality" (p.79). Edwards (1998) supports that "microworlds have been described as computational environments that "embody" or "instantiate" some subdomain of mathematics or science, generally using linked symbolic and graphical representation" (p.55) […while] the notion of linked multiple representations has been key in the development of computer-based microworlds (cf. Confrey, 1993; Thompson, 1985, 1987, in Edwards, 1998, p.61). Ainsworth, Bibby, and Wood (2002) also supported "the use of multiple external representations to support learning is widespread in traditional classroom settings and in computer-based environments […such as] DGS packages which allow tables and graphs to be dynamically linked to geometrical figures" (p. 26).

Dynamic geometry systems (DGS) such as the Geometer's Sketchpad (Jackiw, 2001) or any other DGS software are microworlds which according to Straesser (2002)

> "even if the programs differ in their conceptual and ergonomic design, they share: a) a dynamic model of Euclidean school Geometry and its tools (the dragmode); b) the ability to group a sequence of construction commands into a new command (macro-constructions); c) the visualisation of the trace of points which move depending on the movement of other points (locus of points)" (p.65).

They were designed to facilitate the teaching and learning of Euclidean geometry, Algebra or Calculus and can play a fruitful and crucial role in the process of creating and evaluating conjectures which promote students' creativity, and in so doing greatly contribute to developing mathematical reasoning. A significant number of researchers have investigated topics using DGS environments (for example Govender & de Villiers, 2004; Hadas, Hershkowitcz, & Schwarz,2000; Healy & Hoyles, 1999; Hollerbrand, 2002, 2003, 2004, 2006; Laborde, 2003, 2005; Laborde, Kynigos, Hollebrands, & Strässer, 2006; Leung & Lopez-Real, 2002; Marrades, R., & Gutiérrez, 2000; Mariotti, 2000, 2002; Arzarello, Olivera, Paola, and Robutti, 2002; Christou, Mousoulides, Pittalis, Pitta-Pantazi, 2004; Christou, Pittalis, Mousoulides, & Jones, 2005; Patsiomitou, 2008a, b, 2010, 2012a, b, 2014). The activity of solving problems is based on interaction and transformation between different representational systems (e.g, Goldin & Janvier, 1998) of the meaning. Dynamically linked representations simultaneously displayed and performed help students to translate between different representational systems. The ability to interpret a meaning between representational systems (Janvier, 1987) is necessary for students' conceptual understanding in mathematics. Real world images (or digital images) also "are potential representations […and] offer the heuristic part of learning" as they "denote something" (Kadunz & Straesser, 2004, pp.241-242).

A representation can become *active* in a DGS environment as a student acts on it and decides the steps and techniques toward his aim, trying to address a problem. The term *active representation*s is considered in mindful processing of information in which students individually or in collaboration manipulate and interact with the objects and tools in the dynamic environment and construct their knowledge by reflecting on what they have created. As Jonassen (2004, p.60) points out

> "If we want students to be better problem solvers (regardless of problem type), we must teach them to construct problem representations that integrate with domain knowledge. These internal problem representations must be

> coherent (internally consistent) and should integrate different kinds of representations (qualitative and quantitative, abstract and concrete, visual and verbal).

As I have written previously (e,g. Patsiomitou, 2008a, 2012b), students construct mental linking representations as they interact with dynamic linking visual active representations (LVARs) in many different ways which are dependent on the individual student's conceptual and procedural understanding, and how well-developed their thinking competences and processes are.

A student can construct *linking active representations* (Patsiomitou, 2012a, b): (a) When s/he builds a representation (for example, a figure) in order to create a unmodified construction, using software's interaction techniques by externalizing his/her mental approach or generally by transforming an external or internal representation to another representation in the same representational system or another one. (b) When s/he gets feedback from the *theoretical dragging* (Patsiomitou, 2011a,b, 2012a, b, 2014) to mentally link figures' properties so that, because of the addition of properties, subsequent representations stem from earlier ones. (c) When s/he transforms representations so that the subsequent representations stem from previous ones due to the addition of properties.(d) When s/he links mentally the developmental procedural aspects in a process of a dynamic reinvention (e) When s/he reverses the procedure in order to create the same figure in a phase of a dynamic hypothetical learning trajectory or between phases of the same dynamic hypothetical learning path.

There is a lack of comprehension about how the *active representations* provided by the dynamic geometry software is elaborated or processed in the student's mind; whatever insights I can support are included in the results of my research (e.g. Patsiomitou, 2007a, b, 2008a, b, 2010, 2011a, b, 2012a, b, 2013a, b, 2014, 2016a, b). We can gain insights through continuous recording of brain activity and interactivity with these representations. Other important considerations include: how these connections with the students' pre-existing information are achieved; how this connection is linked to the process of learning; how these active connections are modified; and hence the associations between different linking representations are stored in students' mind.

WHAT IS A LEARNING TRAGECTORY

Simon (1995) defined hypothetical learning trajectories as "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (p. 133). A hypothetical learning trajectory is hypothetical "because […it] "is not knowable in advance" (Simon, 1995, p. 135). He used the metaphor of a sailor to explain the difference between a trajectory and a hypothetical learning trajectory:

> "You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your 'hypothetical trajectory'." (pp. 136-137)

In this thoughtful paragraph, I recognized my own experiences with my every year students in class. The way that my students interacted with the pre-prepared material (digital and otherwise) which I had planned for them, changed the whole path we followed, as I added paths to explain something that was not understood or helped students overcome their misconceptions by using a different path. This was the same feeling I had when I read how Clements & Sarama (2004) defined *learning trajectories* as

> "descriptions of children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (p. 83).

Moreover, in their article "Learning Trajectories: Foundations for Effective, Research –Based Education" in section "What, if anything, is "new" in the learning trajectories construct?", Clements & Sarama (2014) discuss what is new in learning trajectories, reporting the common characteristics the learning trajectories have with psychological and educational theories "for example, Bloom's taxonomy of educational objectives and Robert Gagne's conditions of learning and principles of instructional design, information-processing theories, information- processing models, developmental and cognitive science theories" (p.8-9). Among other things, the same authors support the following:

- "Learning trajectories include hierarchies of goals and competencies but do not limit them solely to sequences of skills as many of the earlier constructs did;
- They are not lists of everything students need to learn;
- They describe students level of thinking, not just their ability to carefully respond to a mathematics question;
- They have an interactionalist view of pedagogy;

- A single problem may be solved differently by students at different levels" (Clements & Sarama, 2014, p.9)

The learning trajectory of the current study was hypothetical at the beginning, as I had hypothesized "if and how [the students] would construct new interpretations, ideas, and strategies" (Fosnot, 2003, p. 10) and the path would follow as they worked on the problem. Moreover, the instructional design process was a synthesis of constructivism and discovery learning, as it was my intention: (a) the students to build on their previous knowledge, (b) the teaching and learning process would be supported through mathematical discourse and conceptual understanding and (c) the learning included students discovery ("aha" expressions) and their dynamic reinvention of knowledge under investigation.

The teacher, however, "*expects the children to solve a problem in certain ways; in fact, expectations are different for different children*" (Fosnot, 2003, p. 10). Each new situation in class requires one or more decisions to be reconsidered in order to bring all the students closer to the predesigned goals. The focal point of interest, and subject under analysis, are the students' answers and the way in which they verbally formulate abstract meanings during the solution of a problem situation. The instructional design process of the current study was designed in phases: for example, what I did towards preparing the lesson before the instruction was delivered; what the organized topics were of the learning trajectory; what I predicted regarding the external stimulation delivered by new representational infrastructures in order to create successive stages in the transformation of previously learned material retrieved from the learner's memory etc.

To design instruction, I had to establish a rationale for what has to be learned in order to be successful. I speak of instruction rather than teaching, because "instruction may include events that are generated by a page of print, by a picture, by a television program, or by a combination of physical objects, among other things" (Gagne, Briggs & Wager, 1992, p.3), including instructional technology in an orchestration process, while the teacher plays an essential in selecting the events and sources as well as their subsequent planning and demonstration. On the other hand, instructional technology, as a "systematic application of theory and other organized knowledge to the task of instructional design" (Gagne, Briggs & Wager, 1992, p.20), has opened a window for students, teachers and researchers to discover and investigate mathematical meanings. Merill (2002) provides a conceptual framework for stating and relating the first principles of instruction. According to Merill (ibid.) learning is promoted when "(a) learners are engaged in solving real-world problems; (b) learners pre-existing knowledge is activated as a foundation for new knowledge; (c) new knowledge is demonstrated, applied and integrated into the learner's world"(pp.44-45). He created the following diagram (Fig. 1) to represent his considerations regarding learning; I have adapted it for the needs of my study.

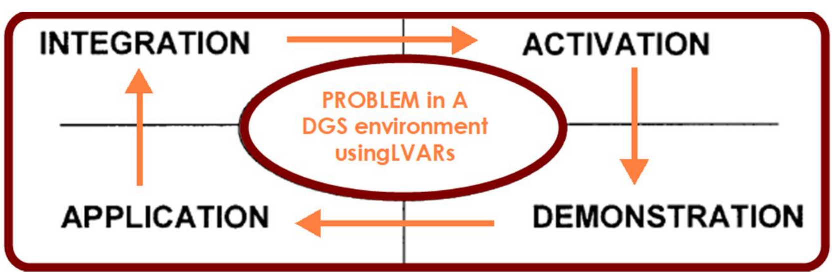


Fig 1. An adaptation for the current study of Merill's (2002, p. 45) phases for effective instruction

Wiggins and McTighe (2005, p.22) also provide a template for design questions addressed to teachers. For example: "What are the big ideas included in the activities? What misunderstandings can be predictable? What learning experiences and instruction will enable students to achieve the desired results?" Also, how will the instructional design help the students to reach the goals and the subgoals of the activity, hold their interest during the process, and provide opportunities to rethink and reflect on their understanding? The main problem in all design principles is according to Gravemeijer (2004) that

> "they take as their point of departure the sophisticated knowledge and strategies of experts to construe learning hierarchies. […] What is needed for reform mathematics education is a form of instructional design *supporting instruction that helps students to develop their current ways of reasoning* into more sophisticated ways of mathematical reasoning. For the instructional designer this implies a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing" (p.106).

METHODOLOGY OF THE EXPERIMENT

The didactic experiment was conducted many times and in many parts through the past years of my career as secondary teacher of mathematics. It is consisted from three phases, described as follows:

Phase A. Firstly, I created the hypothetical learning path for the construction of number pi in March 2006. The students in three classes (almost 75 students, aged 13) participated in the didactic experiment. It was very successful for students' learning and understanding, as a result of which parts of the whole were presented to other teachers of mathematics in cooperation with a school advisor. The most important part of the HLP was the approximation process for number pi through the Geometer's Sketchpad software and the experimental process with a team of students; this is described in the section entitled "Generating number pi in an experimental process". The instruction was conducted in small cooperative teams; I also served as action researcher.

Cobb & Steffe (1983) enhance the idea of the researcher as teacher supporting that "the activity of exploring children's construction of mathematical knowledge must involve teaching". According to Cobb and Steffe there are three important reasons for this: (a) "the insufficiency of relying solely on a theoretical analysis […]; (b) the experiences children gain through interactions with adults greatly influence their construction of mathematical knowledge […]; (c) the importance we attribute to the context within which the child constructs mathematical knowledge […]"(p.84).

Students interacted with the activities designed in the dynamic geometry software, proposed conjectures, and explained results on screen. The trajectory is explained in details as follows (Patsiomitou, 2006a, b, 2007a):

(a) Construction of a square, a pentagon, a hexagon, an octagon and a 17-gon, all inscribed in the same circle using the custom tools provided by the software. I used the same color for the shape of every polygon and its respective calculations to make the connection clear. The polygons were hidden as well as the calculations. An action button was linked to every calculation and the corresponding shape which helped the students to visualize a result in an approximation process for number pi (Fig. 2).

Then I asked the students in the class to calculate the ratio of every inscribed polygon's perimeter to the diameter of the circle.

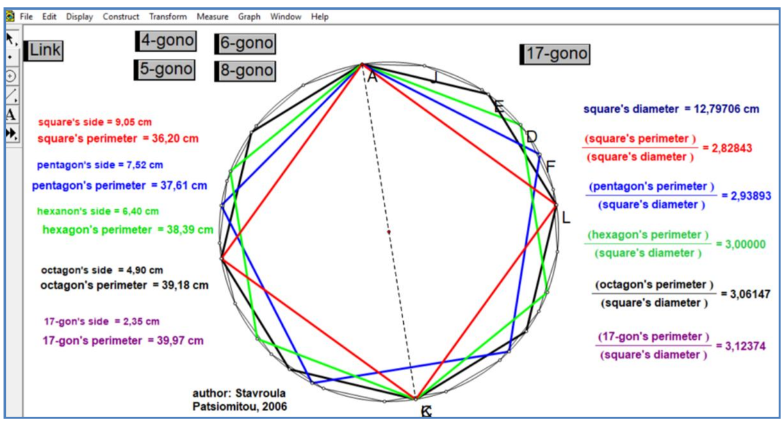
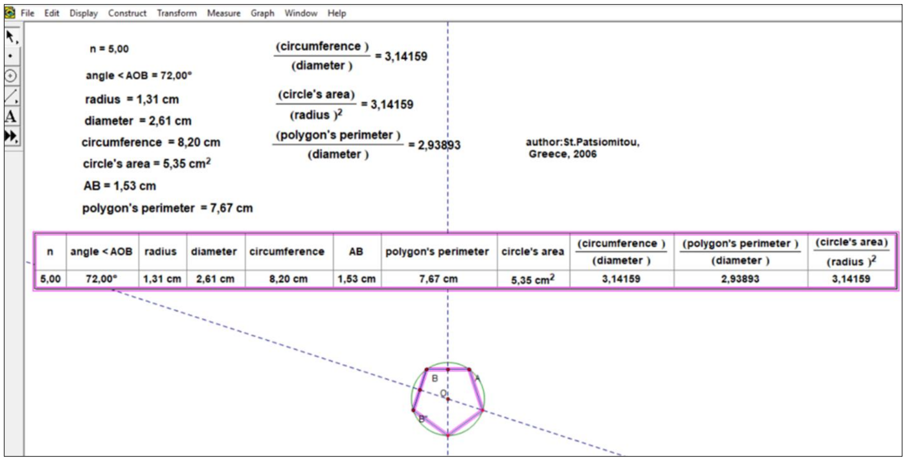


Fig 2. Constructing several polygons and colored links with the respective calculations

(b) The experimental instruction continued using a linked page of the software. The concrete process was a generalization of everything I discussed above, linked in sequential pages of the software. Scott Steketee (2002) in a multiple page sketchpad file presented at the NCTM Annual Meeting (Session 491) conference had constructed among others an n-gon, whose side remained unmodified as the number n was increased. The n-gon can be constructed using the iteration process of the software (e,g., Steketee, 2002, 2004; Patsiomitou, 2006a,b, 2016a, b). In a previous paper I argued that (Patsiomitou, 2007b):

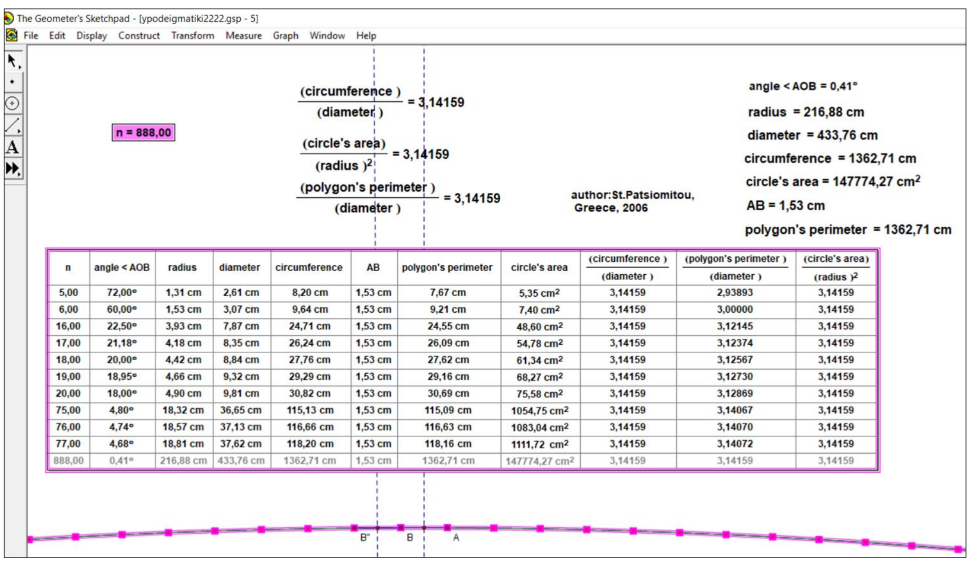
> "*through the process of iteration we can visualize the construction becoming more complex being in theory rendered inductively to infinity. The result of the process of iteration is the construction of the tables that repeat the process of the initial measurements and calculations in dynamic linked representations with the diagram, thus increasing (or decreasing) the level of the process of iteration while the software adds (or removes) the next level of measurements (or even calculations), whereas in the first column of the table the sequence of the natural numbers is presented (in our case the number n represents the number of n-gon's sides)*".

For the instructional process, I created a custom tool with which my students could create an n-gon similar to the one demonstrated by Scott Steketee. The evolution of this process is the following: We construct a polygon inscribed in a circle with n sides, define n as a parameter so it can be increased or decreased depending on the modification the user tries. For example, in Fig. 3 we can view an n-gon with 5 sides along with the tabularized measurements and calculations (i.e. the number of sides of the polygon, the measurements of the radius, the diameter, the n-gon's sides, the n-gon's perimeter, and the ratio of the n-gon's perimeter to its diameter). In Fig.4 we can view an n-gon with 888 sides along with the tabularized measurements and calculations. The length of the side remains unmodified, whereas the iteration process changes the number of sides. As we can see in the final column of the table in Fig. 3, if the number of sides is 5 the ratio of the n-gon's perimeter to its diameter is almost equal to 2, 93. If we continue increasing the number of sides (for example, to 96 or 888 sides) we will see that the approximation process results in a number that is almost equal to number pi. In this process, the length of the side is not affected as the number of iterations increases. The students confirmed the number pi on the table for (n) first steps of iterative constructional steps. My aim was for the students, by means of appropriate questions, to conceive of the circumference of a circle as a convergence of a regular n-gon inscribed or circumscribed using the iteration process. By the end of the lessons, the groups of students had learned how to create the custom tool and the process required generating pi. A few of them had become expert with the software and loved using custom tools and the iteration process.



n	angle < AOB	radius	diameter	circumference	AB	polygon's perimeter	circle's area	(circumference) / (diameter)	(polygon's perimeter) / (diameter)	(circle's area) / (radius)2
5,00	72,00°	1,31 cm	2,61 cm	8,20 cm	1,53 cm	7,67 cm	5,35 cm2	3,14159	2,93893	3,14159

Fig. 3. Links between a parameter, a polygon and its tabularized measurements in an active representation



n	angle < AOB	radius	diameter	circumference	AB	polygon's perimeter	circle's area	(circumference) / (diameter)	(polygon's perimeter) / (diameter)	(circle's area) / (radius)2
5,00	72,00°	1,31 cm	2,61 cm	8,20 cm	1,53 cm	7,67 cm	5,35 cm2	3,14159	2,93893	3,14159
6,00	60,00°	1,53 cm	3,07 cm	9,64 cm	1,53 cm	9,21 cm	7,40 cm2	3,14159	3,00000	3,14159
16,00	22,50°	3,93 cm	7,87 cm	24,71 cm	1,53 cm	24,55 cm	48,60 cm2	3,14159	3,12145	3,14159
17,00	21,18°	4,18 cm	8,35 cm	26,24 cm	1,53 cm	26,09 cm	54,78 cm2	3,14159	3,12374	3,14159
18,00	20,00°	4,42 cm	8,84 cm	27,76 cm	1,53 cm	27,62 cm	61,34 cm2	3,14159	3,12567	3,14159
19,00	18,95°	4,66 cm	9,32 cm	29,29 cm	1,53 cm	29,16 cm	68,27 cm2	3,14159	3,12730	3,14159
20,00	18,00°	4,90 cm	9,81 cm	30,82 cm	1,53 cm	30,69 cm	75,58 cm2	3,14159	3,12869	3,14159
75,00	4,80°	18,32 cm	36,65 cm	115,13 cm	1,53 cm	115,09 cm	1054,75 cm2	3,14159	3,14067	3,14159
76,00	4,74°	18,57 cm	37,13 cm	116,66 cm	1,53 cm	116,63 cm	1083,04 cm2	3,14159	3,14070	3,14159
77,00	4,68°	18,81 cm	37,62 cm	118,20 cm	1,53 cm	118,16 cm	1111,72 cm2	3,14159	3,14072	3,14159
888,00	0,41°	216,88 cm	433,76 cm	1362,71 cm	1,53 cm	1362,71 cm	147774,27 cm2	3,14159	3,14159	3,14159

Fig. 4. Generalization of the measurements in the active representation

During the experimental process, I also used a similar n-gon sketch which resulted in an n-gon with an increasing number of sides, whose length could be modified, circumscribed in a circle. This construction is simple and can be described as follows (e,g., Patsiomitou, 2016 a, b): In a new file in the GSPv4/5 environments, we construct a circle and place a point A on the circumference. We create a parameter n = 5 (random choice of the value '5' that expresses the number of sides of a regular polygon) and then an angle f = 360 ° / n, selecting the parameter n. We select the point A on the circle and rotate it around the center O and the angle f (Fig. 5, 6, 7, 8). We choose the parameter and the (+) sign on the keyboard and increase the repetitions. Then we construct a table, selecting the parameter n and the table generated for successive values of n. Using this construction method, the number of sides of the n-gon increases while the length of the sides of the n-gon decreases.

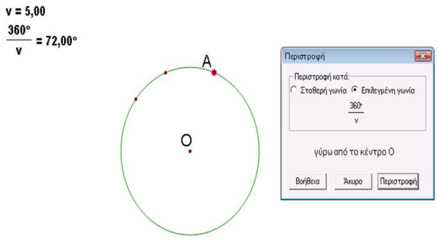


Fig. 5. Creating a parameter n

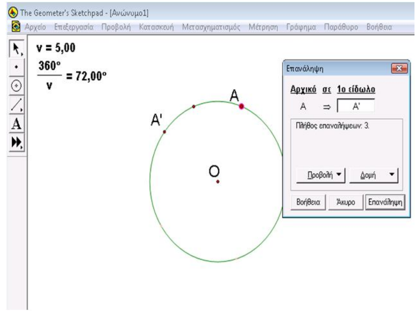


Fig. 6. The instrumental decoding of the iteration process

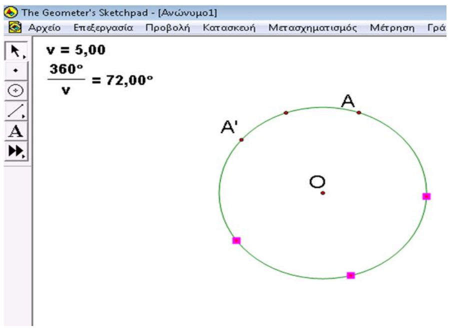
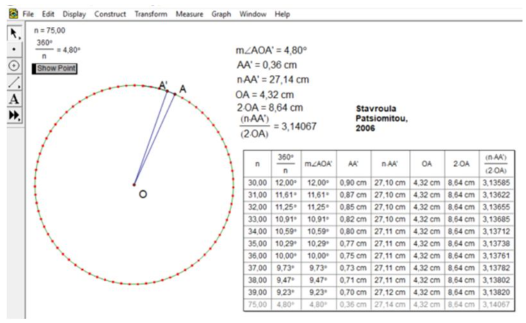


Fig. 7. The iteration process



n	360°/n	m∠AOA'	AA'	n·AA'	OA	2·OA	(n·AA')/(2·OA)
30,00	12,00°	12,00°	0,90 cm	27,10 cm	4,32 cm	8,64 cm	3,13585
31,00	11,61°	11,61°	0,87 cm	27,10 cm	4,32 cm	8,64 cm	3,13622
32,00	11,25°	11,25°	0,85 cm	27,10 cm	4,32 cm	8,64 cm	3,13655
33,00	10,91°	10,91°	0,82 cm	27,10 cm	4,32 cm	8,64 cm	3,13685
34,00	10,59°	10,59°	0,80 cm	27,11 cm	4,32 cm	8,64 cm	3,13712
35,00	10,29°	10,29°	0,77 cm	27,11 cm	4,32 cm	8,64 cm	3,13738
36,00	10,00°	10,00°	0,75 cm	27,11 cm	4,32 cm	8,64 cm	3,13761
37,00	9,73°	9,73°	0,73 cm	27,11 cm	4,32 cm	8,64 cm	3,13782
38,00	9,47°	9,47°	0,71 cm	27,11 cm	4,32 cm	8,64 cm	3,13802
39,00	9,23°	9,23°	0,70 cm	27,12 cm	4,32 cm	8,64 cm	3,13820
75,00	4,80°	4,80°	0,36 cm	27,14 cm	4,32 cm	8,64 cm	3,14067

Fig. 8. Generating number pi

We can continue to experiment and see that the ratio takes a value approximately equal to pi (3, 14067). In this process, the length of the side is affected (decreased) as the number of iterations increases. My students also created their own constructions of iterated n-gons. This process is presented in the "The experimental process using static means" section. A few activities also followed (i.e. a process to allow the students to discover the formula for the area and circumference of a circle, a real problem which was an implementation of the meanings they had learned through the process). The last one related to Proposal 2 in the XII Book of Euclid's Elements--"*Circles are to one another as the squares on their diameters*"--which the students could *instrumentally decode* (Patsiomitou, 2011a, b) into a figure on the screen and prove empirically using measurements and calculations provided by the software (Fig. 9.).

The research questions for this process were the following:

A. Can the students construct the concept of the circumference of a circle as a convergence process of a regular inscribed or circumscribed n-gons' perimeter using dynamic geometry software and the iteration process?

B. Can the students acquire a strong intuition of the process generating number pi as a limit through the iterative ratio (the circle's circumference to the circle's diameter) of the inscribed or circumscribed n-gons in a circle?

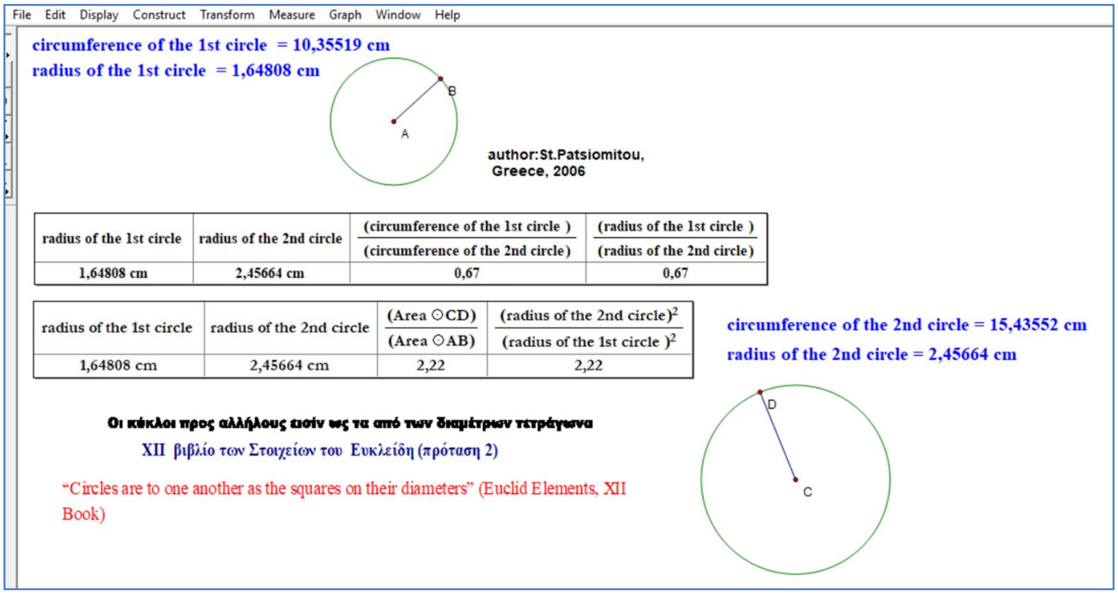


Fig.9. "Circles are to one another as the squares on their diameters" (Euclid Elements, XII Book)

Phase B. Scott Steketee (2004) created among others a sketch presented at the annual meeting of the NCTM conference. The sketch demonstrates that the approximation process of the Riemann sums (for example the Midpoint sum) of a function tends to become a value very close to the calculation of the function's definite integral in an interval AB (Fig. 10), which can be modified by dragging the endpoints A, B of the segment AB. The interval had been subdivided in n subintervals, which were directly linked to a parameter n, which is linked to an iteration process. The increments are parametrical, meaning they can be modified by direct manipulation or by animating the parameter n. The modification of the increments results in the modification of the number of rectangles, as well as the calculation of the Upper, Lower or Midpoint Sum. The Upper Riemann Sum (or the Lower Riemann Sum) uses the maximum (or the minimum) value of the function (respectively) over each sub-interval as the value to accumulate. The rectangles in this sketch are based on such a minimum or maximum value. As we increase the number of sub-intervals, we can take a better approximation of the Riemann Sum. As a consequence, the result of this approximation process is the sum of a sequence of the modified rectangles in sub-intervals equivalently modified in the interval AB for the calculation of an approximation of the definite integral. In Fig. 10 below the Riemann sums of the function $f(x) = ax^3 + bx^2 + cx + d$ have been calculated. We can modify the values of the coefficients a, b, c, d by dragging. This modification results in the modification of the graphic representation, as well as the calculation of the Riemann sums.

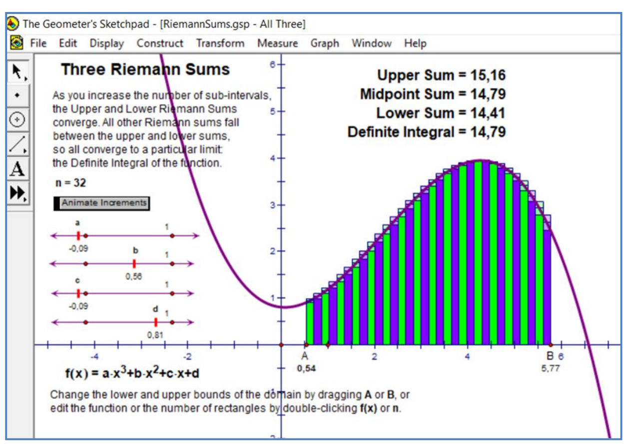


Fig 10. The three Riemann sums and the Definite integral of the function f (Steketee, 2004)

As an implementation of this process (Fig. 11) I similarly created a sketch for the calculation of the definite integral of the function f(x) = $\sqrt{1-x^2}$

The graphic representation of the function f is a semicircle. The area of a semicircle with radius almost equal to 1 is almost equal to 1, 57.

I used the concrete sketch for a case study, as I considered it very interesting. The student was almost 13 years old. My aim was for the student: (a) to use the dynamic representations to approximate the definite integral using the Lower, Midpoint and Upper Riemann Sums and the possible biggest mistakes in these processes, (b) to understand the meaning of the definite integral as an area, and (c) to understand the meaning of number pi as a limit.

If we tabulate and then animate the parameter n, the result is an *active representation* on screen of the linked values of the Riemann sums. This process highlights what is linked in the process and how these modifications affect the tabulated calculations (Patsiomitou, 2005, 2006c, 2007a, b).

Finally, my aim was the students to discover the meaning of number pi through the experimentation process with the Riemann sums.

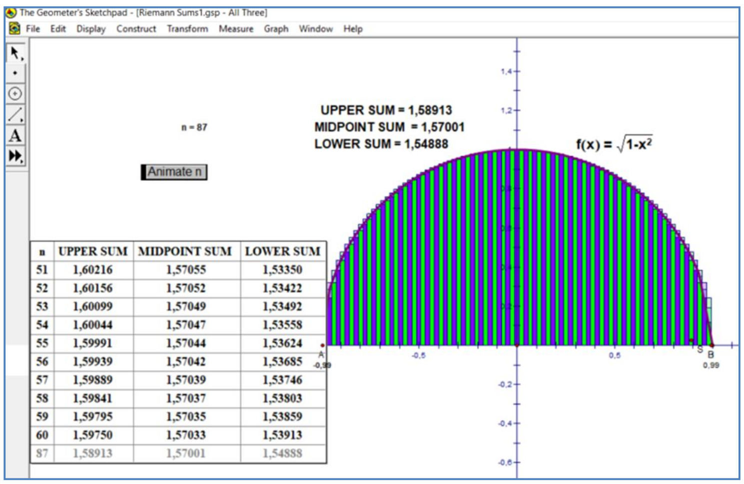


Fig.11. Generating number pi through an approximation process

The research questions of the concrete process were the following:

A. Can the students construct the meaning of a definite integral as an approximation process?

B. How do the students conceive the meaning of pi through the concrete process?

An excerpt from the concrete experimental process is described in the section entitled "The Riemann Sums experimental process". The participating student was almost 13-years-old and very curious about the process for approaching number pi.

Phase C. One more experimental process occurred when I was inspired from the top roof of the Guggenheim Museum, in New York as I was turning Google Earth software. Until the exact moment I didn't know that the roof of the Museum was a regular polygon (a 12-sided regular polygon). This part of experiment is described in the section "The Google Earth approximation of number pi".

According to Merrill (2002) "*Learning is promoted when learners are required to use their new knowledge or skill to solve problems*" (p.49). He also highlighted the importance of many opportunities for instruction as Gardner (1999) did also. However, before I proceed, I will explain what I consider mathematics to be, repeating what Kaput & Roschelle (1999) state:

> "Mathematics is both an *object* of understanding and *a means* of understanding. […]While our universe of experience can be apprehended and organized in many ways – through the arts, the humanities, the physical and social sciences – important aspects of our experience can be approached through systematic study of patterns. Such aspects include those that are subject to measure and quantification, […] that involve our place in space and the spatial features of the world we inhabit and construct, and that involve algorithms and more abstract structures (p.155)"

The concrete process can be considered an application of what my students had learned already. Moreover, to understand that mathematics are part of our lives and can interlink with many other disciplines. I designed the following successive stages (Patsiomitou, 2016a, b): (a) I prompted my students to use Google Earth to collect data about the places we intended to explore (e.g. a place in the United States). In this way, the students would be responsible for understanding the differences and similarities identified between a place in the US and their own country. The purpose for this step was for the students to be gradually led to conclude, for example, that: all places in the world are composed of mountains and plains; that they are divided into smaller regions and states which have a capital; that life in cities may not differ from life in Athens or any European capital. A second task was for my students to use the zoom tool in the Google Earth software to stimulate their interest in investigating sociological content further—for instance: what are the buildings like in the US; what kinds of cars are there; how are the people in the streets dressed, etc. Part of this stage was to find a building with mathematical interest in order to explore the mathematical meanings. Such a building is the roof of the Guggenheim museum. (b) The next task was to create successive images which would help them to create linking representations in their mind (Patsiomitou, 2012a, b, 2014) of the objects under investigation and to recall the information involved in a new learning object. Moreover, the students to create the modeling process in the DGS environment and understand the meaning of pi as an approaching process (as reported in Phase A). Schumann (2004) created a diagram that "presents an outline of methods and ways of working with DGS in the context of geometry teaching in lower and middle secondary schools; modeling in DGS is supported by all other methods and options"

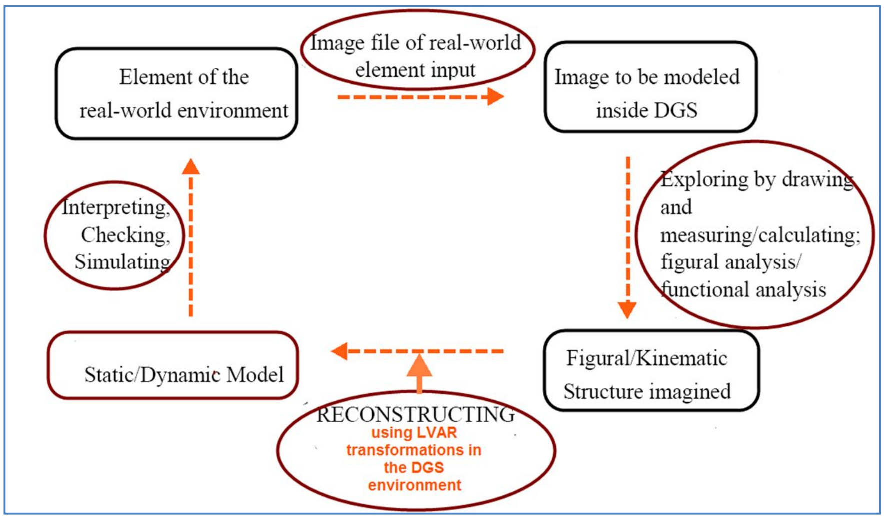


Fig.12. An adaptation for the current study of Schuman's (2004, p. 7) modelling process using LVAR

For the representation of a theoretical diagram using tools and theoretical constructs I introduced a pseudo-Toulmin's model (Patsiomitou, 2011, 2012a, b) --based on Toulmin's model (1958)-- in which: (1) the data could be the dynamic diagram, or an object and (2) a warrant could be a tool or a command that guarantees the result which is the claim (or the resulted formulation). In the Fig. 12, the factor "reconstructing using LVAR transformations" is the warrant (W) for the claim (static /dynamic model). This means that LVAR transformations guarantee the interpretation of the dynamic model to students' mind into external [verbal or iconic] representations.

The experimental process

Part A: Generating number pi in an experimental process

We started with the construction of a circle with diameter 1, so that its circumference was equal to number pi. The students then constructed an inscribed regular pentagon, hexagon, octagon, 17-gon etc. in the same circle, and calculated the ratio of the perimeter of the polygon to the diameter of the circle for every polygon. I guided them to tabulate the values and visualize what happens to the ratio when the number of sides increases. Then they investigated the parametric regular n-gon and tabulated the results of the central angle of the polygon, the length of the side of the polygon, etc. as described in the methodology section. The students tried many modifications of n (for example n=3, 4, 5, . . . 50 . . .) and then calculated the length of the diameter and perimeter of the circle. Then they calculated the ratio of the circumference of the circle to the diameter of the circle and visualized the modifications that appeared on screen as they changed the parameter n of the polygon's sides. In this way, they could view the approximating process for generating number pi.

All the students (13–14 years-old) were encouraged to talk about what they were doing, to describe what they could view, and to make conjectures about the procedure. Here is an excerpt from our discussion in class with the students M1, M2, M3:

Researcher: *What do you observe?*
M1: *I noticed that the length of the circle is almost equal to the perimeter of the polygon…*
M2: *The circle's circumference tends to become equal with the length of the polygon's perimeter (i.e. 118,20)*
M3: […] *as we increase the number n the lengths tend to become equal. (i.e. 118,20)*
M3: *We notice that the n-gon's perimeter and the circle's length tend to become equal, and that any time one increases the other increases too, though the length of the circle is always a little bigger.*
Researcher: *What about the ratio L/d?*
M1: *If we increase the number n this number (*showing on the screen*) approaches 3.14 ..(3,14067..3,14072…3,14159)*
M2: *The result is almost equal to number pi…*
M3:*If the number of the polygons' angles increases then the number of the quotient approaches number pi.*

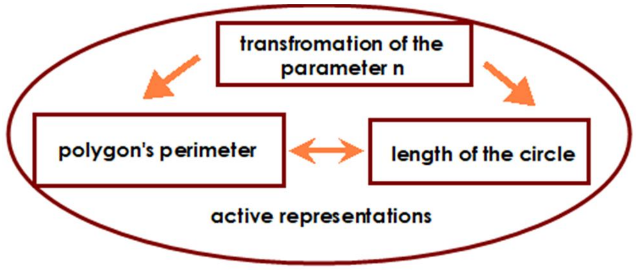
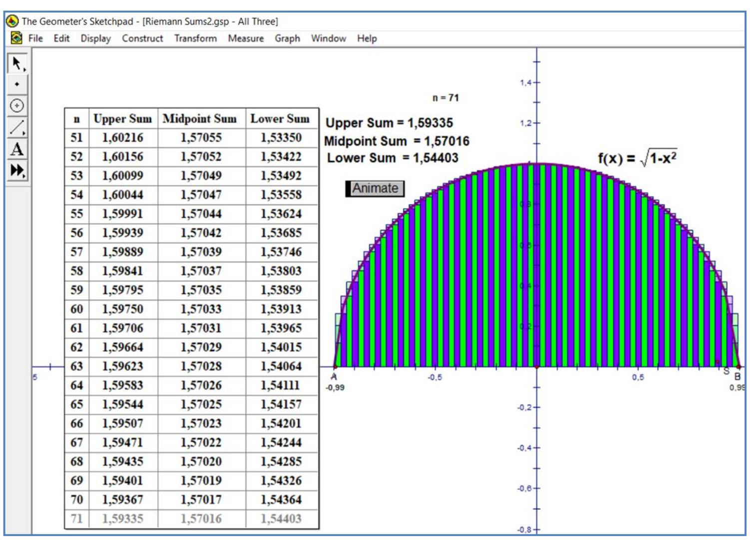


Fig.13. Transformations and active representations

Part B: The Riemann Sums experimental process
The experimental process and the dialogue with the student M5 went as follows:
Researcher: *What can we view when we increase number n?*
The student experimented with different values of n and answered:
M5: *We observe that if we increase number n the number of sub-intervals increases (to be exact, she said "the number of columns") and the width of every column decreases. The columns start from Point A and continue to Point B. Moreover, if the columns become thinner, they touch the line of the function.*
Then she observed that the way that the graphic representation of the function is created correlates with the way the software uses the Riemann sums, going on to note that:
M5: *In the Lower Sum, the function is on the left side of the rectangle. In the Upper Sum, it is on the right side of the rectangle. In the Midpoint Sum, it is in the middle.*

For the function f(x) = $\sqrt{1-x^2}$, the student observed that the graphic representation is a semicircle at the interval [-0.999.., 0.999..] (Fig. 14).



n	Upper Sum	Midpoint Sum	Lower Sum
51	1,60216	1,57055	1,53350
52	1,60156	1,57052	1,53422
53	1,60099	1,57049	1,53492
54	1,60044	1,57047	1,53558
55	1,59991	1,57044	1,53624
56	1,59939	1,57042	1,53685
57	1,59889	1,57039	1,53746
58	1,59841	1,57037	1,53803
59	1,59795	1,57035	1,53859
60	1,59750	1,57033	1,53913
61	1,59706	1,57031	1,53965
62	1,59664	1,57029	1,54015
63	1,59623	1,57028	1,54064
64	1,59583	1,57026	1,54111
65	1,59544	1,57025	1,54157
66	1,59507	1,57023	1,54201
67	1,59471	1,57022	1,54244
68	1,59435	1,57020	1,54285
69	1,59401	1,57019	1,54326
70	1,59367	1,57017	1,54364
71	1,59335	1,57016	1,54403

Fig 14. The Riemann sums of the function f

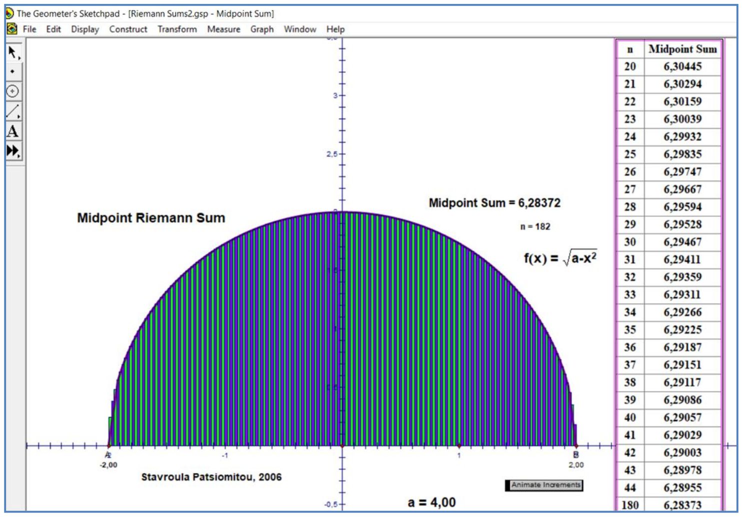


Fig 15. The Midpoint Riemann sum of the function f

In the case that the value of the radius of the circle is equal to 0.9999≈1, she observed that Ls = 1, 54403, Ms = 1.57016, Us = 1, 59335. Moreover, she ordered these values and formulated that "*Ls < Ms <Us even if we change the values of parameter n*". Then I asked her to calculate the area of the semicircle (with radius r=1). She answered that it was almost 1, 57 and spontaneously highlighted the number with a circle.

> **M5:** *As n is increased, the values are modified*: *in the Lower Sum the values increase, in the Upper Sum the values decrease. […] all the values tend to equal 1,57, which is the value of the area of the semicircle*
>
> **M5:** [Then she continued] … *meaning that the values coincide. I think that the software* ***calculates the area of the semicircle.***

The student had never heard anything to this effect; it was a discovery, since she *dynamically reinvented* a meaning.

Then I changed the type of the function f(x) = $\sqrt{a-x^2}$

The more general type of the function did not challenge the student because she replaced the parameter a with the number 4 (Fig. 15). She observed that the tabulated values were tending to the number 6, 28 (when the parameter a=4). She noted:

> **M5:** *The value 6,28 must be the area of the semicircle. […] Yes […] it is the area of the semicircle. A ha*!. *Since we know the radius is constant and the area is modified but tends to a number […]* ***then we can understand why number pi tends towards 3, 14*** *(which we already know from our books).*

The student is an isolated paradigm, and the method is a case study; we cannot generalize the results of the method. What the research process evoked can only be characterized as primary results. The student did not have experience with the software or with any other related packages. The approximation process guided her to reach a conclusion about the order of the Riemann Sums. Standing alone, the calculations could not help her verify her intuition. The tabulation of the measurements helped her, as it was a part of *active linking representations*. Meaning that the dynamic diagrams linked with the linking parameters and then with the tabulated sequential successive measurements. These were *dynamic linking visual active representations* which helped the 13-years-old student to connect many meanings and formulate using deductive argumentation. Moreover, the easy deletion and repetition of the tabulation for different parameters led the student to observe that the approximation of the number pi ≈3.14 was repeated for random/incidentally or continued values. Most importantly, the student acquired a very strong intuition regarding the meaning of the definite integral. When I asked her "what do you think is a definite integral? Can you give a definition? M5 answered: "hmm…***it is an area …but we have two dimensions so we speak of an area …if we had three dimension would we have a volume?"***

The experimental process using static means

This part consists of the experimental process with two students using static means: Part A describes the process with a high-achieving student and Part B with a low- achieving one.

Part A. The 60-gon in Fig. 14 is a work of a 13-year-old male student who used the software (Patsiomitou, 2006e). The assignment I gave them was: "Can you create a polygon with n-sides? For example, can you construct a 17-gon, a 20-gon etc". Many students created figures using static means (for example cardboard constructions). The most impressive were two assignments by two students. I will present here the construction of a 60-gon in the student's own words: *"In* *order to construct a 60-gon, I first constructed a circle with a random radius. Then I used the rotation transformation to transform Point A on the circle by 6 degrees to the right of Point A. Point B had now been constructed. I constructed the chord AB, which is also the side of my polygon. Then I used the iteration process to iterate the process 60 times, constructing in this way a 60-gon. In order to color the figure, I used the command 'construct triangle interior', choosing the interior of the triangle and rotating it by 12 degrees*" (Fig. 16).

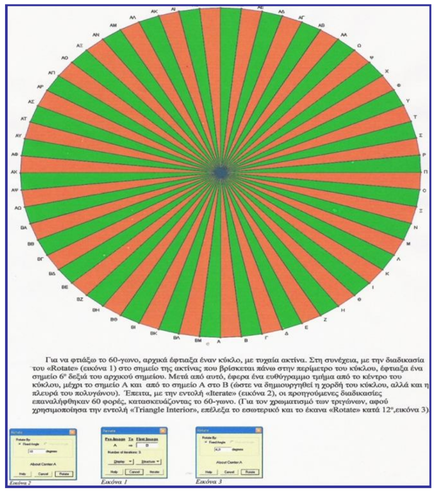


Fig 16. My student's construction and description in the software

So he continued the process. I helped him to use the iteration process to color the figure in an easy way (Fig. 20). In the Fig. 17, 18, 19, I repeat what my student did for the construction with a little help from me.

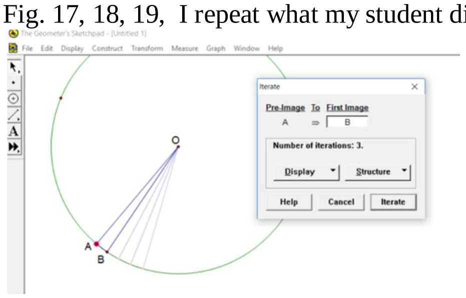


Fig. 17. Rotation and iteration of point A

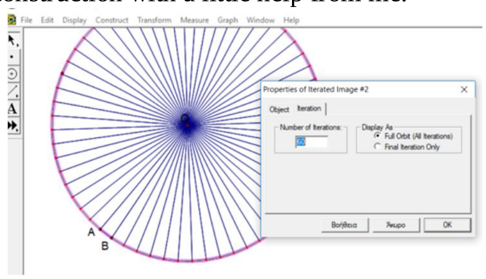


Fig. 18. Construction of a 60-gon

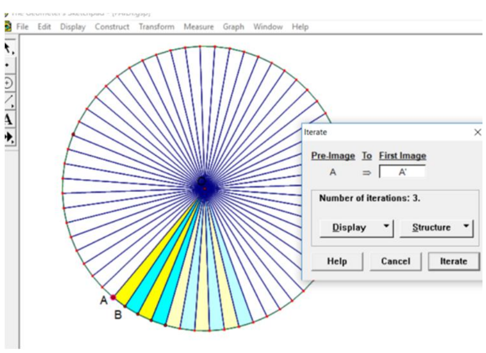


Fig. 19. Coloring the 60-gon using rotation

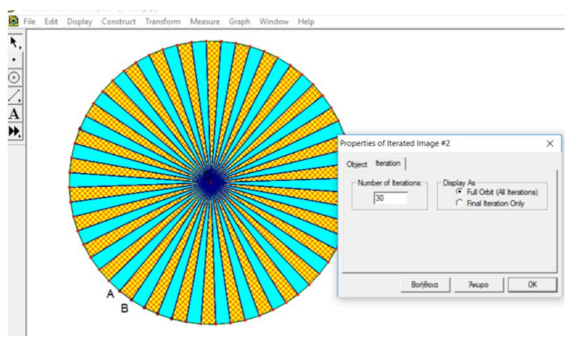
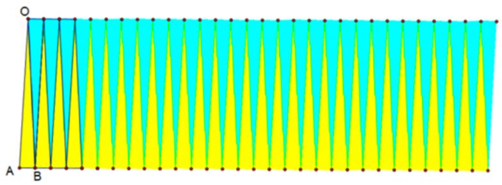


Fig. 20. Coloring the 60-gon using the iteration process



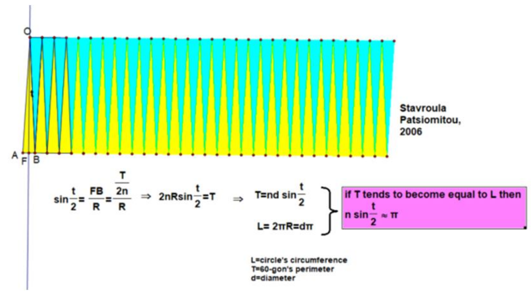


Fig. 21. Reconfiguration of the 60-gon

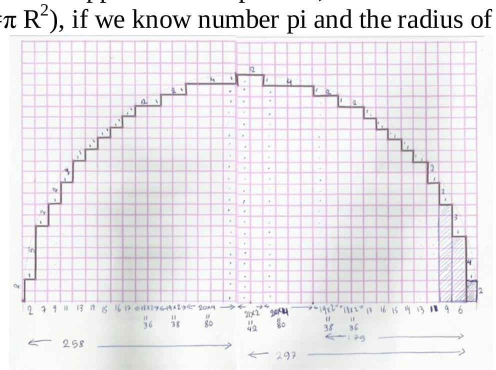
Fig. 22. Deductive argumentations

He continued the process, explaining the whole procedure. He described how he created the 60-gon. He used the iteration process and explained how he used the software's pop-up menu in detail. This is what I would later call *instrumental decoding* (Patsiomitou, 2011a, b), meaning *the way that an individual transforms his/her mental representations into an external representation on screen using the software's instrumented techniques*. He also showed me how he developed it on screen visually through his operational apprehension of the figure, a *reconfiguration* according to Duval (1999), and verbally through his detailed description. It should be noted that this student participated in the expert team (6–7 students, all 13 or 14 years old) with which I collaborated, and whom I was giving instructions in using the software, for an hour after the lesson in school. Then he tried to construct a rectangle with the triangles of the inscribed 60-gon. "*I shall square the circle* (he laughed) [...] *Firstly, we will copy and paste the triangle AOB of the inscribed polygon on the screen*. *Then we will translate this image triangle by vector AB*" (Fig. 21).

The student could also repeat the construction in an orchestration process with the whole class. Then I helped them to understand why $E=\pi R^2$. "*The figure is a rectangle so* $E= (L/2) R= \pi R^2$". My students were very enthusiastic about the process above. I grasped the opportunity to introduce them to an important element of deductive argumentations in a symbolic way (Fig. 22), showing them how they can correlate the number pi with the angle t (central angle of the inscribed polygon). The important point is that the most of students participated in creating the symbolic representations, an indication that they had developed their thinking.

Part B. As an action researcher, every time I phase difficulties with my students' understanding, I use assessment tasks or post-tests to view how they have understood meanings after I have used digital technologies to present them. The artifacts used to "metaphorize" mathematical processes of concepts are constructed out of materials which are either static (a card construction, for example) or digital (a dynamic geometry software construction). In this sense, abstract meanings can become concrete through the figurative material.

Here is an example of a student's work on squared paper; the student in question, M4, is 13, had difficulties understanding mathematics, and liked to create figures with Geometer's Sketchpad. He had not heard about the Riemann sum's approximation process, but he did know about the formula with which we can calculate the area of a circle ($E=\pi R^2$), if we know number pi and the radius of the circle.



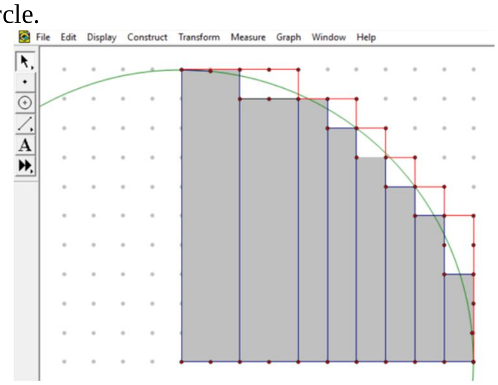


Fig. 23. Calculating the area of a semicircle

Fig. 24. Calculating the area of a quarter circle using the software's facilitations

I prompted him to construct a circle on the squared paper, and then asked him "*Can you calculate the area of the circle using the squared paper*?"

The squared paper was not enough to construct a semicircle, so he constructed two quarter circles and he joined them with glue. Obviously, he did not consider calculating the area of quarter circles from the beginning. I reminded him of the limitations of the study:

Researcher: *Let us suppose that we do not know the number pi and we would like to calculate the area of the circle. What could we do about this*?

M4: *We could count all the smaller squares and then we would sum them. This will give us the area of the circle.*

Researcher: *OK . . . you can do that. Can you think of a way to find number pi*?

M4: *Yes, we could divide the area of the circle by the radius squared… hmm.*

M4: (After the process) [...] *we have a semicircle so we can duplicate the number 555.* (Fig. 23)

I prompted him to continue the process and he found that the number pi was almost equal to 3, 425... He was not satisfied as he knew that number pi is almost equal to 3,14.

M4: *We don't need a semicircle to calculate the number pi. We can repeat the process twice. Can we continue on the software*? (Fig. 24)

He constructed a circle on a quarter sheet of paper he had pasted into the software, but when he arrived at the calculation of number pi, he observed that the construction was not accurate, so he tried again on the software's dot_paper, instead. He created a polygon circumscribing and another inscribing the circle and continued with the same process. Using this method, he found that the number pi was almost equal to 3.04. But he did not stop there. He told me:

M4: "*We have to add these squares that are outside the circle. The area of the circle lies between these values*".

He did the calculations and found a number almost equal to 3, 3.

M4: "*Now we can understand why the value of number pi is almost equal to 3, 14. This number is between the numbers 3, 04 and 3, 3, as the line of the circumference's circle is between the lines of the polygons outside and inside the circle"*.

The student was unable to make connections between the tables and the graphs, but he followed this step-by-step process. This was a discovery for the student, as I think he found a little piece of knowledge: a process for estimating number pi.

Although part A describes the process followed by a high achievement student in mathematics and part B describes the process followed by the low one, I should mention that the students' answers seem to belong to the same level. This is a result of their interaction with the software, which scaffolds the students' interpretation of the meanings presented through its visual dynamic active representations.

Part C: The Google earth approximation of number pi

In the concrete the virtual earth 'flew' on Earth in class, displayed satellite images as it was rotating, and stopped at a region of the United States where we could observe and grasp a visual library of information regarding the place (for example, the US and the oceans surrounding it)--in other words the geographic features of the environment. Therefore, students first acquired an overview of the geophysical position of the US on the world map. In Fig. 25 we approached the US using the zoom navigation and approach tool. The students could perceive at first glance the number and location of the different US states, answering scaffolding questions regarding their position on the map, their capitals, and the human and physical characteristics of the places, cities' connections with nearby towns, and focusing their attention on a city on the map (for example, the position of New York in the US in Fig. 25). Students could observe the city and the layout of the buildings, as well as major urban centers in the city, districts made famous by the cinema, such as Brooklyn, New Jersey and Astoria. We could also zoom in to read street names and famous bridges, and attempted to descend into Broadway. They then focused their attention on the Guggenheim museum, whose aerial view is very interesting from a mathematical point of view.

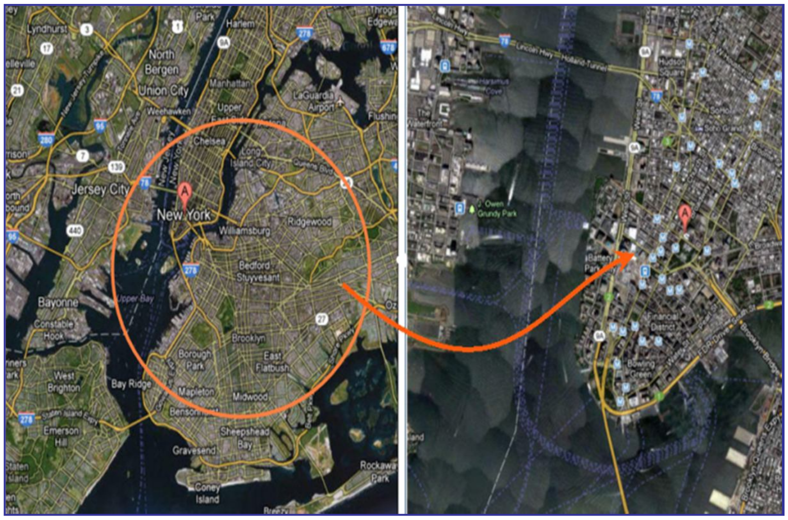


Fig.25. Using Google Earth to make discoveries about life in the US

We copied and pasted the image onto a dynamic geometry software sheet the Geometer's Sketchpad (Fig.26, 27, 28, 29) in order to solve a real-world problem (e.g. what kind of regular polygon does the floor plan of the museum represent, what are the real dimensions of the scale figure, which figures can be constructed if we join the diagonals of the 12-gon, what figure can we create if we construct lines perpendicular to each radius of the 12-gon etc.).

Oldknow (2003) also used “the Geometer’s Sketchpad to illustrate an approach to algebraic modeling applied to a digital photograph of a fountain […for this] he pasted them to show how Sketchpad always treats this […] ” (p.16), something I did also later following him (e.g., Patsiomitou, 2013b, 2014, 2016a, b). As Oldknow states “ […] both the most recent versions of the […] most widely used dynamic geometry software packages allow you to perform both geometric and algebraic modeling using captured images” (p. 19).

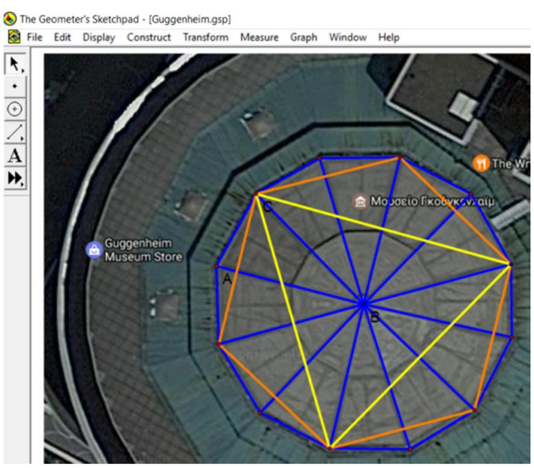


Fig. 26. Copy and paste the top view into the Geometer’s Sketchpad

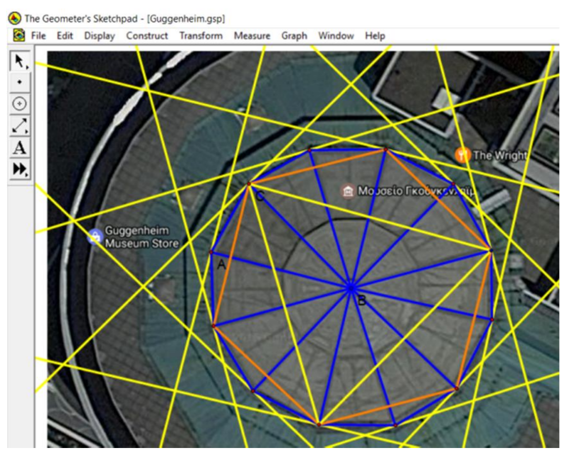


Fig.27. Figures constructed joining the diagonals of the 12-gon

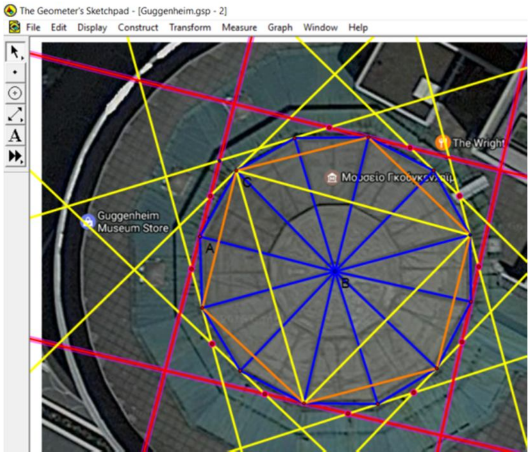


Fig. 28. Constructing regular polygons using the Geometer's Sketchpad

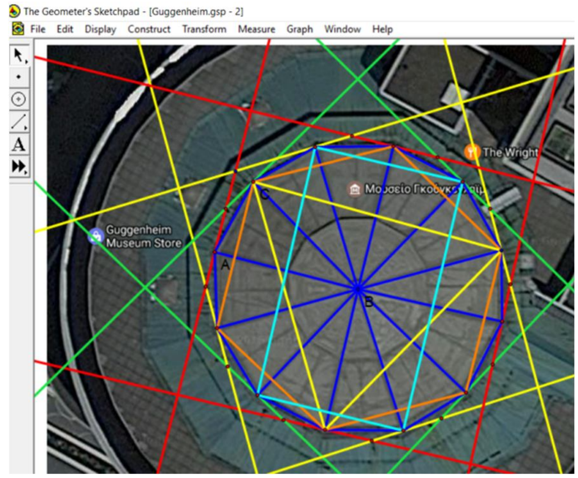


Fig. 29. Solving several problems using the figure

In the illustrations above we can view the top view of the Guggenheim museum and a depiction of its appearance produced on Geometer's Sketchpad v4.5. Also, the figures demonstrate the constructions of regular polygons using the Geometer's Sketchpad v4/5. Using the Web to teach mathematics can provide an abundance of activities to enhance your everyday lesson. I used one such activity, which I found at the web page of the NCTM, to challenge my students; an interactive math applet, it allowed me to connect the area of geometry with the area of algebra in mathematics (Webpage [3]). The real-world problem relates to the handshakes a group of 12 people exchange when they meet each other (Fig. 30). The specific questions were these: How many handshakes occur between the 12 people? And, in general, how many handshakes occur when there are n people? The activity is interactive and the students could view the sequence of numbers on screen by pressing a button, as well as the lines connecting the vertexes in the figure. They could also create a pattern how the next number that would be displayed on screen.

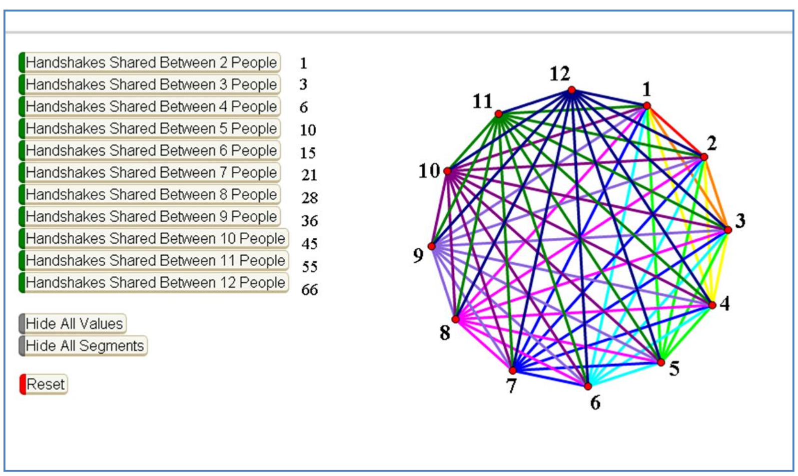


Fig.30. Screenshot from the handshakes math applet (NCTM, webpage [3])

Here is an answer reached in cooperation with the students: "*We have 12 people, so the first person shakes hands 12 times. The other 11 people will each shake hands 12 times, which equals 132 handshakes*. *However, it takes two people for a single handshake, so the answer is half of 132, meaning 66 handshakes.*

RESULTS AND DISCUSSION

The role of *linking visual active representations* (e.g. Patsiomitou, 2008a, b, 2012a, b) was very crucial to understand pi as an estimating process. As a consequence, the investigation process regarding the number 'pi' helped the students to use the outcomes of their previous processes in order to make connections and to develop a generalization. They created their own demonstrations in the DGS environment, reinventing processes, formatting concepts and transferring their ideas using dynamic and static means. In the diagram below (Fig. 31), the warrant is the iteration process which transforms the parameters and then the figure and the tabularized measurements into linking representations. The concrete process becomes *an active representation* which affects students' thinking. The result of the inductive process is an abstract mathematical object.

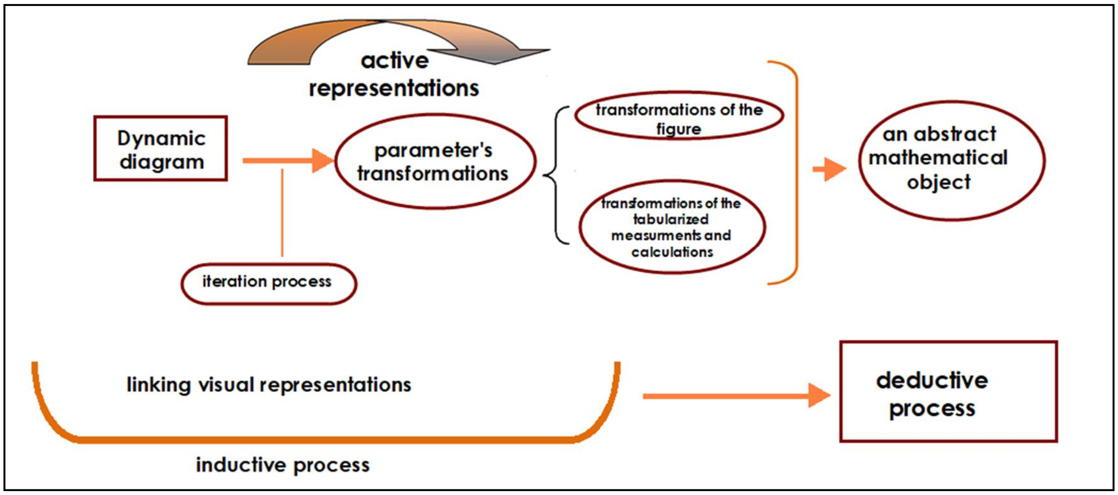


Fig.31. Transformations using active representations

The understanding of mathematical meanings is connected with the potential to conceive them simultaneously as objects and as processes. As Sfard (1991) claims *an interaction between a process or an object is indispensable for a deep understanding of mathematics whatever the definition of "understanding" may be.* The proposed approach leads the students to consider number pi as an object but they do not miss the sense of the process that leads to the object (Monaghan, Sun & Tall, 1994). I designed the learning trajectory to investigate the links between conceptual and procedural knowledge while it is evolved directly as an *instrumentalisation* process among the mediated artefacts (computer software and the related activities), the various school level groups or the isolated students interacting and the action researcher. All these tools which had been integrated in the learning process enabled students to mathematize via abstraction the didactic phenomena and structure a successive process of procedural and conceptual meanings though a dynamic trajectory, as a result of the linking representations created in their mind. It reflects on Kaput's (1992) writing who argues that calculus concepts may well be introduced into middle education curricula. Also, what Kaput & Roschelle stated (1999, p.164) "*Computational media are reshaping mathematics, both in the hands of mathematicians and in the hands of students as they explore new, more intimate connections to everyday life*". This arises such questions as: what curricular ideas are appropriate for what grade level and even what curricular ideas are appropriate for certain ability tracked classrooms? When and how should complex concepts such as Riemann sums and definite integrals be introduced to mathematics students?

Papert (1984) in his paper "Microworlds: Transforming Education" describes the experience of a little girl who discovered number "zero" as she played with a microworld. This was a crucial point for her understanding, as she understood that the command "S0" made the microworld stop moving. As Papert argues (1984, p. 81)

> "I think she was excited because she had discovered zero. They tell us in school that the Greek mathematicians, Pythagoras and Euclid and others, these incredibly inventive people, didn't know about zero. […]The fact that not every child discovers zero this way reflects an essential property of the learning process. No two people follow the same path of learnings, discoveries, and revelations. ***You learn in the deepest way when something happens that makes you fall in love with a particular piece of knowledge***."

These words of Papert made me think of my own process with my students. They loved this particular piece of knowledge with its active representations that made different students discover number pi in several different ways, at different times over the years. I also fell in love with the particular incidents, which have played an important role in my thinking process since then. The role the active representations play in the learning trajectory which, though it may take several different routes to reach it, has the same learning goal, made me think of a way to define what a dynamic active learning trajectory is, based on the previous definitions of Simon (1995) and Clement & Sarama (2004, 2014): **Dynamic Active Leaning trajectories** are sequential instructional tasks and activities engaged in [with] a learning goal and designed [with dynamic active representations] to engender mental linking representations which help students develop their thinking in the specific math domain.

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