Primary school pupils' strategies for mental addition and subtraction computations

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Abstract
Performing mental calculations is a complex metacognitive activity that relies on the knowledge and basic skills of the learner, on the strategies he/she uses, and on his/her ability to refer to how he/she works, how he/she processes and how he/she recovers data from their cognitive system. The aim of the research is twofold; a) investigate the strategies used by pupils aged 8 to 12 for the effective calculation of additions and subtractions of two-digit numbers and, b) study pupils' evolution, as they grow older. The findings highlight a wide variety of strategies for the effective calculation of addition and subtraction using only the mind, both high-level and low-level, with the mental representation of the written algorithm, the separation and the accumulation strategies being the most preferable. Moreover, the study provides insight into the representational content and the way information is organized in memory.

Keywords: Mental calculations, Strategies, Implementing decisions, Additions, Subtractions, Computations

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1. Introduction

Mind calculation is a complex mental process involving the activation of a frontal-brain network of brain regions (Vansteensel et al., 2014) for the conscious numerical calculation of a result. As an activity is accomplished without the use of tools (Reys, 1984), it begins with the computational manipulation of small numbers, as the result of the mind process, and either through formal education or age, it evolves to higher levels, differently for each person. Effective mental calculations is a complex metacognitive activity that relies on the knowledge and basic skills of the individual, the strategies he uses (Maclellan, 2001) and his ability to tell how he works, how he processes and how he retrieves data from his cognitive system. It requires flexibility and adaptability, while it is linked to the deeper development of the sense of numbers and the understanding of the structure, relations and properties of the numerical system (Torbeyns & Verschaffel, 2013). Flexibility is defined as the ability to switch between different tools to find solutions (Rathgeb-Schnierer & Green, 2013). Adaptability emphasizes on choosing the most appropriate strategy to calculate a correct result (Verschaffel et al., 2009). The term sense of number refers to the general understanding of the numbers and functions of a person along with the ability to use this understanding flexibly in managing numerical situations (McIntosh, Reys, & Reys, 1992). In children the sense of mental calculations is developed naturally in their early years of age. After the age of four or five, by entering school life, they learn how to manage their minds in more formal ways (Varol & Farran, 2007). However, some researchers argue, that children can develop their own effective and specialized strategies, spontaneously (Carpenter et al., 1998, Kamii et al., 1991).

In recent decades, the development of flexible mental calculations was considered as a significant goal for elementary school (Lemonidis, 2016). In his effort to point out the importance of intellectual computations compared to writing calculations, Maclellan (2001) pointed out that while the use of written algorithms encourages children to follow different steps without thinking about what they are doing, mental calculations allow them to participate in the process defining the importance of numbers in the problem. Mental calculations promote the development of problem-solving skills, provide techniques for calculating estimates, contribute to understanding the concept of number, and encourage children to manage numbers with ease, establishing the foundation for developing computational skills. Furthermore, mental calculations help pupils understand and evolve written computational methods (Karantzis, 2011). According to the international literature (Cobb and Merkel, 1989; Heirdsfield & Cooper, 2004a; Maclellan, 2001; Sowder, 1992), mental counting of multiple digits insertions and abstractions is crucial in teaching children to make decisions about procedures and to reinvent different strategies for solving mathematical problems. Gürbüz & Erdem (2016) mention that regular practice on mental calculations contributes to the development of pupils' strategies, ability to explain, critical ability and helps cultivate the sense of numerical and procedural functions. Reys et al. (1995) underline the importance of these conceptual calculations, their usefulness in everyday life, and their contribution to the promotion and monitoring of a higher level of mathematical thinking. It has been observed that children novice in mental calculations ignore even the obvious numerical properties that would help them, rendering them entirely dependent on the reproduction of standard algorithms on paper (Sowder, 1992).

Since all the above underline the importance of numerical computations, the next section aims to explore the strategic mental calculations used by elementary school pupils to perform only with mind regarding the mathematical operations of addition and subtraction. As opposed to traditional problem-solving techniques, numerical calculations include a wide range of strategies.

The purpose of this paper is to investigate the strategies employed by pupils aged 8 to 12 in order to calculate two-digit numbers in additions and subtractions using only the mind. This is important to explore becausestrategies reveal aspects of the ways the child uses to effectively perform the operations of addition and mind subtraction (executive decisions). Furthermore, the study of such a mental activity provides information about the representational content of the strategies, the way of organizing the information in memory and, finally, its evolution alongside with the pupil's aging maturity.

2. Strategies of addition and subtraction mental calculations

In recent years, the influence of cognitive psychology on the study of mental processes has been significant as it has broadened the horizon of the study of human behavior by incorporating in its scope the cognitive processes involved in solving mathematical problems. Mental calculation strategies have been characterized by many researchers as the specific cognitive processes, actions or behaviors that are deliberately implemented to develop individual solving skills. According to Thompson (1999), it is the application of known or quickly calculated numerical data in combination with special properties of the numerical system in order to find the solution of a calculation whose answer is unknown. Mental calculations are learned, intent, goal-directed and controlled modes of action (Kostaridou-Efklides, 2011a).

According to the Theory of the Mind, metacognitive declarative knowledge refers to the awareness of the mental state for the cognitive self, the others as cognitive beings and the cognitive processes for the relations with the various cognitive works, goals, strategies and the related experiences (Kostaridou- Efklides, 2011b). In particular, the Theory of the Mind as a representation (Perner, 1991) refers to the way the cognitive system of the child works, how it processes , how the data is retrieved and how it manages information. It specifically examines why, how and in which case a person applies these strategies. All these processes occur through monitoring which, as declarative knowledge, implies the representation of being aware. The knowledge of the strategies for recalling and storing new information in memory constitutes a declarative representation of the strategy, yet it differs from its implementation, which is procedural knowledge. Likewise, monitoring as declarative knowledge differs from control for the latter requires the combination of both procedural knowledge and executive decisions.

Mental calculations strategies are classified as acts, depending on their nature, of instrumental or relational understanding (Callingham, 2005). On the one hand, the conductive/contributing strategies are those that are implemented without any explanation indicating the understanding of the underlying concepts. Otherwise, correlative understanding indicates the conceptual understanding of numbers. Reviewing the literature, a variety of strategies are found to be employed by students when they perform numerical calculations for addition and subtraction with two-digit numbers (Blote, Klein, & Beishuizen, 2000; Heirdsfield & Lamb, 2005b, Karantzis, 2011, Lemonidis, 2013; Macintyre & Forrester, 2003; Van de Walle, 2007). These are:

The declarative knowledge consists of a specific body of numerical factors from which the solution of basic numerical problems can be resolved immediately and without effort (Karantzis, 2004; Wolters et al., 1990). Direct recall (AUTO) refers to this retrieval of the result of a numeric execution from long-term memory, which does not bear a large cognitive load, e.g. 10 + 10 = 20, 76-76 = 0, with the person responding directly to the stimulus of the act.

The mental representation of the typical addition or subtraction algorithm is a strategy in which the individual mentally expresses the procedural knowledge, places the numbers one under the other as he/she would do on paper and processes the mathematical operation, from right to left, for example. 54 + 29: 9 + 4 = 13, I hold 3 and carry one, 5 + 2 = 7 and one equals 8, hence 83.

The strategy of separating tens and ones (1010) is considered one of the most common strategies for the addition and subtraction of natural numbers and is so called for the reason that numbers used for the mental operation are divided into multiples of ten and ones (Heirdsfield & Lamb, 2005a). It appears in two forms, either as a separation technique from left to right (e.g. 41 + 26: 40 + 20 = 60, 6 + 1 = 7, 60 + 7 = 67 and 87-36: 80-30 = , 7-6 = 1, 50 + 1 = 51), or as a separation technique from right to left (u-1010, e.g. 71 + 24: 4 + 1 = 5, 70 + 5 = 95 and 54-32: 4-2 = 2, 50-30 = 20, 20 + 2 = 22). Nevertheless, this strategy does not seem to be convenient in subtraction, for the reason that when the ones of the subtrahend number are greater than that of the minuend, i.e. when we need to transfer. According to Varol and Farran (2007) this strategy also includes the mixed separation-accumulation technique (10s) where, for addition, the ones of the first additive are added to the sum of the tens, followed by the ones of the second adder e.g. 36+ 44: 30 + 40 = 70, 70 + 6 = 76, 76 + 4 = 80). Regarding the subtraction with a digit of transfer, the numbers are broken in their tens. Then by the difference between the tens, the ones of the subtrahend are removed and added to the ones of the minuend, (e.g. 79-27: 70-20 = 50, 50-7 = 43, 43 + 9 = 52). Finally, in subtraction with one transfer digit first takes the dozens, and then the ones of the minuend are added to this difference and by this result the ones of the subtrahend are removed (e.g. 96-39: 90-30 = 60, 60 + 6 = 66, 66-9 = 57).

The accumulation strategy (N10) or alternatively the jump-method is based on the reason that the first term remains stable while the second is broken down into ones and tens to be added or subtracted from the first one. It occurs in two formats, either as the technique of tens-ones accumulation (e.g. 17 + 81: 17 + 80 = 87, 87 + 1 = 88 and 62-45: 62-40 = 22, 22-5 = 17) or as accumulation of ones-tens (u-N10, e.g. 81 + 19: 81 + 9 = 90, 90 + 10 = 100 and 83-53: 83-3 = 80, 80-50 = 30). According to Karantzis (2011), the emergence of this strategy also occurs when the second term of the operation is broken down for jump by dozens, e.g. 47 + 26: (47 + 10 + 10) + 6 = 73 and 85-34: (85-10-10-10) -4 = 51. According to Lemonidis (2013), this strategy is given the name “jump” because it can be easily represented, empirically or mentally, in a number line where you start from a number and move to the answer by jumping onto the line by adding or removing appropriate parts of the second number.

The holistic strategies are usually employed by learners who have developed mind flexibility, since they require complex mental manipulations (Lemonidis, 2016). Based on the techniques that stem from holistic number management, students make additions or subtractions with the given numbers in order to reach more operable numbers, namely numbers can be added or subtracted more easily (Lemonidis, 2013; Van de Walle, 2007). These particular techniques are commonly applied to numbers that are close to ten. One applicable technique is that of compensation (N10C) where one of the two numbers is rounded up or down in order to reach the nearest ten (e.g. 47 + 35: 50 + 35 = 85, 85 - 3 = 82 and 98 - 45: 100 - 45 = 55, 55 - 2 = 53). It can also be considered as an accumulation strategy (N10) moving to the nearest ten. Another technique for implementing a holistic strategy is that of balancing (A10 or C10). This technique resembles the previous one, but differs where we add or subtract a certain number to the first number of the operation, we add or subtract the same number to the second number, too, so that the two interventions are mutually eliminated and have a final result of zero or equilibrium [$\alpha + \beta = \gamma$, $(\alpha-\kappa) + (\beta + \kappa) = \gamma$)], e.g. 59 + 34: 59 + 1 = 60, 34-1 = 33, 60 + 33 = 99.

The reason behind the numbering strategy deals with the step-by-step counting of a sequence of numbers (Geary et al., 2004), starting from the first term and going up or down so many steps as the second term of the addition or the subtraction shows (CON), either by using the fingers (COF), or by the rhythmical movement of the head (COH, Lucangeli et al., 2003). It is found in the form of two techniques, numbering with ones (e.g. 24 + 12: 24 + 1 = 25, 25 + 1 = 26 ... 27, 28, 29 ... 36 and 99-7: 99-1 = 98 ... 97 , 96, 95 ... 92) and numbering with tens (e.g., 33 + 24: 33 + 10 = 43, 43 + 10 = 53,53 + 1 = 57).

The completion of the subtrahend is used only in subtractions and is implemented with two techniques, depending on the distance that separates the minuend from the subtrahend. The student increases the ones of the subtrahend until he/she reaches the minuend; the correct answer is the number that represents the increase. When the distance between the two numbers is small, the student sums the ones one by one, or all the ones together to the subtracted until he/she reaches the minuend (e.g. 91-87: 87 + 4 = 91). When the distance between the two numbers is large, the final answer appears through steps where the tens are first and then come the ones (e.g. 85 - 28: 28 + 2 = 30, 30 + 50 = 80, 80 + 5 = 85, 2 + 50 + 5 = 57).

Finally, indirect subtraction (IS), a strategy encountered exclusively in subtractions. It is executed when we start from the subtracted and gradually remove until we reach the minuend with the correct answer being the number representing the reduction e.g. 76 - 28 : 76 - 6 = 70, 70 - 40 = 30, 30-2 = 28, 6 + 40 + 2 = 48. According to Lemonidis (2016) this strategy is rather rare and appears to be more effective when there is little difference between the terms of subtraction.

According to Karantzis (2011), the numbering strategy (CON, COF, COH), as well as the mental calculation through the mental representation of the typical algorithm (MA), are characterized as low-level strategies since they are inefficient and do not promote mathematical thinking in children (Heirdsfield & Cooper, 2004b, Sowder, 1992). On the contrary, high-level strategies contributing to better understanding of numbers are the strategy of separation (1010), accumulation (N10), holistic strategies, add-on strategy and indirect subtraction (IS). They are considered more effective and tend to alter according to the numbers involved, to facilitate calculation (Van de Walle, 2007). Thus, they contribute to the ability of a person to adapt their choice of an appropriate strategy to calculate a correct result (Verschaffel et al., 2009).

3. Method

3.1. Participants

A total of 320 elementary school pupils, 150 boys and 150 girls coming from the region of Central Greece and Crete took part in this research. The distribution of pupils per class was equal for better comparison of results. Of the 300 participants, the 80 students were enrolling the 3rd grade, 80 students the 4th grade, 80 students the 5th grade, and finally 80 the 6th grade.

The students were selected randomly from a wider sample of 8-12 year old students who participated in mental tests to standardize the sample by gender and age categories. The age range of pupils enrolled is considered appropriate for the purpose of the research as children of middle childhood age have developed the ability to report on their beliefs about their cognitive activities (Berk, 2015). Also, for the selection of the participating students in the research, a number of criteria were imposed: a) children with serious developmental problems or learning difficulties or sensory or cognitive disorders were excluded from the sample, and (b) there were included children who had responded correctly and had justified their response to all tests in mental calculations of addition and subtraction.

3.2. Research Questions

The research questions formed for the investigation of the purpose of the study are the following:

(i) Which strategies are more frequently employed by students aged 8-12 years to effectively calculate additions and subtractions of two-digit numbers?

(ii) How does the strategy of mentally calculating additions and subtractions of two-digit numbers evolves as the pupil grows?

3.3. Procedure

The participants of the study took part in a series of tests of mental calculations regarding the operations of addition and subtraction with a result from 20 to 100. The tests consisted of eight operations, of which half were additions (26 + 23, 42 + 27, 69 + 34, 24 + 58) and the remaining half were subtractions (57-25, 79-43, 84-28, 91 -69). Two of each of the operations did not include a carry, while the other two required a carry.

Each student was asked individually to calculate only by their mind the result of the sequence of operations, presented to them in a horizontal order, to discourage the use of traditional algorithms (Karantzis, 2011; Van de Walle, 2007). Then, the students were asked to write down, by performing each mental calculation, the way they thought (think-aloud method). The main advantage of the written externalization is that the student's reports are generated almost simultaneously with the thinking processes and reveal the solution procedures as well as discloses the cognitive evidence included in this process.

3.4. Analyses

The data gathered from the think-aloud records were analyzed qualitatively according to the model of Miles and Huberman (1994). 2560 reports emerged for the 300 participants. Data was undergone a first level reduction in order to be converted into codes, which held a functional definition and were classified into broader categories. The second level of reduction was to group the stemmed categories and codes into common final themes, which are conceptually identifiable. The codes, the categories and the themes were then presented in summary tables (Merriam, 2009). Thereafter, the categories were grouped into the thematic axes that were created beforehand concerning the strategies and techniques employed for the calculation of (a) mental additions and (b) mental subtractions in the form of frequency and relative frequency tables (see Results). Given that there was differentiation regarding the quality and/or accuracy of the students' reasoning, in order to ensure reliability in the categorization of strategies and techniques, the degree of agreement between researchers with the Cohen's kappa (k) affinity index was checked, where a high degree of reliability ($k = 0.99$) was identified.

The numerical data obtained from the qualitative analysis were transferred to linear array tables in the SPSS 20 statistical packet and were analyzed quantitatively. The χ^2 homogeneity test was applied to answer the research questions. A level of statistical significance (p) was set at 5% whereas the findings with $p < .05$ were considered statistically significant. The magnitude of the effects was calculated using the Cramer's V index (Fritz et al, 2012, p. 12).

4. Results

4.1. Mental calculations in additions

Regarding the use of the strategies employed to reach a correct result in the tests of addition operations, a variety of strategies were observed with more frequent the mental representation of the written algorithm (413 references). As one of the participants mentions for the operation 24 + 58, *"four plus eight equals twelve, two plus one carry, five plus one six plus two ... eight, so eighty two"* (student 138).

With regard to high-level strategies, the most preferred is that of separation (403 references). The main techniques for implementing the separation strategy (1010), with respect to its representative content, is that of the separation from right to left (26 + 23: *"six plus three ... nine, twenty plus twenty ... forty, thus forty-nine"*, student 47), the separation from left to right (42 + 27: *"forty plus twenty... sixty, seven plus two ... nine, so sixty-nine"*, student 81), but also its combination with the strategy of accumulation (24 + 58: *"fifty plus twenty ... seventy, seventy and eight ... seventy-eight and four, eighty-two*", student 268).

Table 1 Frequencies of appearance of addition mental calculations strategies per class

Categories	Codes	Codes' meaning	References per grade				N	N%
			3rd	4th	5th	6th		
Low Level	MAPS	Mental algorithm representation	115	93	96	109	413	32,27
	CONP	Numbering strategy	51	56	45	27	179	13,98
High Level	1010S	Separation strategy	87	103	99	114	403	31,48
	N10S	Accumulation strategy	63	62	61	48	234	18,28
	HOLS	Holistic strategies	4	6	19	22	51	3,98

The strategy of accumulation (234 references) appears to follow in frequency the separation strategy, with its implementation techniques recorded to be the accumulation of tens-ones (69 + 34: *"sixty nine plus four ... seventy three, seventy three plus thirty ... one hundred three"*, student 174) and the accumulation of tens-ones (69 + 34: *"sixty nine plus thirty ... ninety nine, ninety nine plus four... one hundred three"*, student 251). The numbering strategy appears 179 times, either through the technique of numbering ones (e.g. 26 + 23: *"twenty-six plus one twenty-seven ... plus one, twenty-eight ... plus twenty-nine… forty-nine"*., student 184), either by numbering tens (e.g. 26 + 23: "*twenty-six plus ten ... thirty-six ... plus ten, forty-six ... and three, forty-nine*", student 9).

Finally, holistic strategies are observed in a few cases (51 reports) mainly by the pupils' enrolling the last grades of primary school. Holistic approaches emerge through compensatory techniques (69 + 34: *"I turn sixty-nine to seventy ... seventy plus thirty-four, one hundred four, plus one ... one hundred three*", student 298), balancing techniques (26 + 23: *"I turn twenty six into twenty five, so, twenty-five and twenty-three... twenty-five, fifty, fifty, one minus..that makes forty nine"*, student 265), and techniques that move to the nearest ten (A10 or C10, e.g. 24 + 58: ... *"fifty eight plus two... sixty, sixty plus twenty-two, eighty-two"*, student 277).

Regarding the adoption of each strategy, based on the homogeneity check on the frequencies recorded per class, it is revealed that the numbering strategy is gradually abandoned as the children grow older [x^2 (3) = 10.74, p<.05, Cramer's V = .24]. Also, the frequency of references to holistic strategies seems to increase in the higher classes [x^2 (3) = 19.35, p <.001, Cramer's V = 0.33]. On the contrary, the frequencies of the strategy of mental representation of the written algorithm [χ^2 (3) = 3.18], the strategy of separation [χ^2 (3) = 2.54] and the strategy of accumulation [χ^2 (3) = 2.55] do not differentiate statistically significantly (p> .05).

In additions without a carry there were reported a total of 215 low-level strategies, of which 64 in the third grade, 54 in the fourth, 50 in the fifth, and 47 in the sixth grade. Accordingly, 425 high-level strategies were classified with a frequency reference of 96 for the third grade, 106 for the fourth, 110 for the fifth and 113 for the sixth grade, respectively (see Table 2).

Table 2 Frequencies of high level and low level mental calculation strategies for addition per class, for operations with or without a carry

Operation	Strategy level	Grade 3rd	4th	5th	6th	N
Addition without a carry	Low level	64	54	50	47	215
	High level	96	106	110	113	425
Addition with a carry	Low level	102	95	91	89	377
	High level	58	65	69	71	263

A total of 377 low-level strategies were identified regarding additions with a carry. More specifically, 102 in the third grade, 95 in the fourth, 91 in the fifth, and 89 strategies in the sixth grade were reported. Also, 263 high-level strategies with a reporting frequency of 58 for the third grade, 65 for the fourth, 69 for the fifth and 71 for the sixth grade, respectively, were recorded.

Although the frequency of low-level strategies seems to be reduced both in addition with a carry [χ^2 (3) = 1.05], and in addition without a carry [χ^2 (3) = 3.07] in accordance with the pupils' grade, no heterogeneity was revealed in the distribution of their values (p> .05). Likewise, the strategies of high level strategies of mental calculation using only the mind [without a carry: χ^2 (3) = 19.35, with a carry: χ^2 (3) = 19.35, p> .05]

4.2. Mental calculations in subtractions

Regarding the use of the strategies used to reach the correct result in subtraction calculations, it seems that most of the times the cognitive representation of the written algorithm (644 references) is employed, irrespective of the student's class (eg 84-28: "*four minus, is not possible so fourteen, take out eight, that's six… eight minus three, five… therefore fifty-six*", pupil 51).

Table 3 Frequencies of appearance of subtraction mental calculations strategies per class

Categories	Codes	Codes' Meaning	References per grade 3rd	4th	5th	6th	N	N%
Low Level	MAPM	Mental algorithm representation	185	163	149	147	644	50,31
	CONM	Numbering strategy	19	13	12	9	53	4,14
High Level	N10M	Accumulation strategy	79	82	78	70	309	24,14
	1010M	Separation strategy	37	48	41	36	162	12,66
	ISSTM	Indirect subtraction	0	2	20	22	44	3,44
	ANTM	Add-on removal strategy	0	12	11	18	41	3,20
	HOLM	Holistic strategies	0	0	9	18	27	2,11

Also, the accumulation strategy is used fairly often (309 reports), as seen on Table 3. The main application of the strategy is through the accumulation technique of tens of ones (e.g. 91-69: "*ninety one minus sixty ... thirty one, thirty one out nine ... twenty two*", student 138) and more rarely with the accumulation technique of ones-tens, especially in cases of regrouping (e.g. 57-25: "*fifty-seven, five out… fifty-two… minus twenty... thirty-two*", student 40).

The separation strategy (162 references) is processed through the right-to-left separation techniques (e.g. 79-43: "*nine out of three ... six, seventy out of forty ... thirty ... so thirty-six*", student 6) as well as separation from the left-to the right (e.g. 57-25: "*fifty to twenty ... thirty, seven out five ... two, so thirty-two*", student 18), especially for the cases where the number of ones is less than the number of ones to be deducted, that is, we do not have to regroup.

The numbering strategy (53 reports) is adopted with a small frequency, either by subtracting the ones (e.g. 57-25: *"fifty-seven minus one ... fifty six, fifty-five, fifty-four ... thirty-two"*, student 52), either by subtracting ten by ten (e.g., 79-43: *"seventy-nine, sixty-nine, fifty-nine, thirty-nine, three out... thirty-six"*, student 74) or by combining the two techniques, that is subtracting tens and then ones (e.g. 79-43: *"seventy-nine, sixty-nine, fifty-nine, thirty-nine ... thirty-eight, thirty-seven, thirty-six"*, student 185).

Finally, mainly in the upper classes of elementary school, the strategies of indirect subtraction are rarely employed (44 reports, e.g. 57-25: *"fifty-seven minus seventeen ... forty, forty out three ... thirty-seven, thirty-seven out five ... thirty-two"*, student 312), subtraction through addition (41 references, e.g. 91-69: *"sixty nine plus one seventy, seventy and twenty ... ninety, ninety and one ninety-one, so one and twenty-one equals twenty-two"*, student 318) and holistic strategies (27 references, e.g. 57-25: *"sixty minus twenty equals thirty-five, thirty-five minus three equals thirty-two"*, student 297).

Regarding the adoption of each strategy, the homogeneity check on the frequencies recorded per class reveals that the strategy of indirect deduction [χ^2 (3) = 36.75, p <.001, Cramer's V = .91] , the strategy of subtraction through addition [χ^2(3)=16.47, p<.001, Cramer's V=.*63*] and the holistic strategies [χ^2 (3) = 33.0, p <.001, Cramer's V = 1.0], despite their rare occurrence, increase significantly as children grow older. On the contrary, the frequencies of appearance for the cognitive representation of the written algorithm [χ^2 (3) = 5.71], the numbering strategy [χ^2 (3) = 3.98], the strategy of accumulation [χ^2 (3) = 1.02] and the strategy of separation [χ^2 (3) = 2.20] are not statistically differentiated (p> .05).

With regard to the subtractions by regrouping, also called borrowing or trading, a total of 310 low-level strategies were reported, of which 87 in the third grade, 79 in the fourth grade, 73 in the fifth and 71 in the sixth grade. 330 high-level strategies with a reference frequency of 73 for the third grade, 81 for the fourth, 87 for the fifth and 89 for the sixth grade respectively (see Table 4) were categorized.

Table 4 Frequencies of high and low-level mental calculation strategies for subtraction and subtraction by regrouping, per class

Operation	Strategy level	References per grade 3rd	4rd	5rd	6rd	N
Subtraction without regrouping	Low level	87	79	73	71	310
	High level	73	81	87	89	330
Subtraction by regrouping	Low level	117	97	88	85	387
	High level	43	63	72	75	254

Concerning the mental subtraction by regrouping, a total of 387 low-level strategies were reported, of which 117 in the third grade, 97 in the fourth, 88 in the fifth, and 85 in the sixth grade. Also, 254 high-level strategies with a reference frequency of 43 for the third grade, 63 for the fourth, 72 for the fifth and 75 for the sixth grade respectively were identified.

In spite of the fact that the frequency of the low-level strategies appears to decrease both in the subtraction by regrouping [χ^2 (3) = 6.56], and in the subtraction without regrouping [χ^2 (3) = 2.00], correspondingly to the student's class, as well as an observed increase in high-level strategies for subtractions without a borrow [χ^2 (3) = 1.88], there was no heterogeneity in the distribution of their values (p> .05). On the other hand, the high-level strategies when regrouping are revealed heterogenenous [χ^2 (3) = 9.88, p <.05, Cramer's V = .20], with the frequency values in the higher classes be increased.

5. Implications & conclusions

The purpose of this study was to investigate strategies that pupils aged 8 to 12 years employ in order to effectively mentally calculate additions and subtractions of two-digit numbers. According to the stemming results, a wide variety of strategies eventually emerged as well as light was shed on the way pupils operate the strategies.

The first finding refers to the main strategy of solution for additions and subtractions using only the mind, which seems to be the strategy of the mental representation of the written algorithm. Students imagine that the numbers move and take their place one under the other, to represent a vertical operation (Heirdsfield & Cooper, 2004a). Subsequently, high-level strategies, such as the separation and the accumulation strategy, are largely adopted. The above findings are consistent with the results of similar surveys (Karantzis & Tolou, 2009; Lemonidis, 2013; Karantzis, 2011). The research reveals that Greek students make wide use mainly of separation strategies, although it is noted that several students also use the accumulation strategy. Indeed, the latter is used more effectively than the separation strategy. Also, in a large frequency, not only in Greece but also in other countries of the world, the mental representation of the formal algorithm appears as a strategy of solution with the mind (Lemonidis, 2013; Karantzis, 2011; Macintyre & Forrester, 2003). Studies have shown that separation strategy is used to a large extent in the United States of America, mainly due to the fact that from the first classes of elementary school pupils are systematically taught the division of numbers into ones and tens. On the other side, European countries emphasize on the strategy of accumulation as it is attributed to minimize the percentage of mistakes made by students (Varol & Farran, 2007). Blote et al. (2000) in their study to investigate which addition and subtraction strategies are used by Dutch elementary school pupils concluded that the predominant mental strategy is that of accumulation.

The second finding relates to the constant choice in the use of specific strategies as the age of the pupil advances in order to effectively resolve the operations. Although a drift of inverse relationship exists between the frequency of the low-level strategy employment and student class, both for additions/subtractions with a carry and for additions/subtractions without a carry, this was not verified for the variable of the growing age. Similarly, the employment of high-level addition mental strategies, except for subtractions by regrouping, was more frequent in the higher grades. An exception is also for the mental calculations of the addition, the reduction in the appearance of the numbering strategy as the child is in higher classes and the growth of holistic strategies. In the mental calculations of subtractions, the frequency of indirect subtraction strategies, subtraction through addition and holistic strategies has increased, although they have been recorded to a small extent. The observed stability in selecting several effective strategies at the age of 8-12 can be attributed to multiple factors: a) by the learner’s reliability in making executive decisions that do not require much cognitive load, b) pupils are trained through formal education in picking these strategies first and c) the above mentioned strategies comprise of safe solution options. In terms of differentiation of some strategies into larger classes, these are interpreted by further individual development in cognitive flexibility, adaptability and deeper sense of numbers (Geary et al 2004; Torbeyns & Verschaffel, 2013). The sense of numbers, depending on the acquired level, leads to improved understanding and resolution, with rapid and flexible ways of managing numerical situations. This explains why effective but time-consuming techniques (e.g. numbering techniques) are gradually abandoned and replaced by more complex but fast resolving strategies (holistic strategies).

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